THE EMERTON-GEE STACKS FOR TAME GROUPS, I

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ABSTRACT. We construct the moduli stack of L-parameters for tame p-adic groups and prove their Noetherian (formal) algebraicity.

CONTENTS

1. Introduction

In the categorification program of the local Langlands correspondence ([EGH23], [FS21], [Zhu21]), it is desirable to have the moduli stack of L-parameters. For p-adic and mod p local Langlands, categorification is not merely an aesthetic pursuit, but rather a necessity. For example, there are far more supersingular representations of $\mathrm{GL}_2(F)$ ($F \neq \mathbb{Q}_p$ unramified) than there are supersingular L-parameters for $\mathrm{GL}_2(F)$. While the set-theoretical local Langlands correspondence lacks a p-adic analogue, a categorical/geometric approach has been proposed in [EGH23]. This approach involves the consideration of coherent sheaves on the moduli of L-parameters as the objects of interest on the spectral side.

In this paper, we generalize the construction carried out in [EG23] to tame p-adic groups, and build the moduli of L-parameters over $\mathrm{Spf}\, \mathbb{Z}_p$.

Throughout the paper, G denotes a connected reductive group over K that splits over a tame extension E/K. Write ${}^LG = \widehat{G} \rtimes \mathrm{Gal}(E/K)$ for the Langlands dual group of G, defined over $\mathrm{Spec}\, \mathbb{Z}_p$. We write Gal_K for the absolute Galois group of K.

The main result of the paper is that there exists an fppf stack $\mathcal{X}_{{}^LG} \otimes \mathbb{F}_p$ over $\mathbb{F}_p$ (Definition 7.1.1) satisfying the following:

Theorem 1 (Theorem 10.2.1, Proposition 7.1.3)**.** *(1) $\mathcal{X}_{{}^LG} \otimes \mathbb{F}_p$ is a Noetherian formal algebraic stack over $\mathbb{F}_p$.*

(2) For any Artinian local $\bar{\mathbb{F}}_p$-algebra A, the set of equivalence classes in $|\mathcal{X}_{{}^LG}(A)|$ is in natural bijection with the set of L-parameters $H^1_{\mathrm{cts}}(\mathrm{Gal}_K, \widehat{G}(A))$.

Moreover, if E is tamely ramified over $\mathbb{Q}_p$, then $\mathcal{X}_{{}^LG} \otimes \mathbb{F}$ admits a p-adic thickening $\mathcal{X}_{{}^LG}$.

Theorem 2 (Corollary 10.2.2)**.** *Assume $E/\mathbb{Q}_p$ is tame.*

(1) $\mathcal{X}_{{}^LG}$ is a Noetherian formal algebraic stack over $\operatorname{Spf} \mathbb{Z}_p$*.*

(2) For any Artinian local $W(\bar{\mathbb{F}}_p)$-algebra A, the set of equivalence classes in $|\mathcal{X}_{{}^LG}(A)|$ is in natural bijection with the set of L-parameters $H^1_{\mathrm{cts}}(\mathrm{Gal}_K, \widehat{G}(A))$.

Theorem 2 also holds for many wildly ramified fields E (see Subsection 1.1 for details). We establish the geometric Shapiro's lemma, which allows us to replace K by $\mathbb{Q}_p$.

Proposition 1. *(Proposition 7.2.4) Assume the splitting field E of G is tame over $\mathbb{Q}_p$. There is a canonical isomorphism* $\mathrm{Sha} : \mathcal{X}_{{}^LG} \xrightarrow{\cong} \mathcal{X}_{{}^L\mathrm{Res}_{K/\mathbb{Q}_p} G}$.

The proof of the geometric Shapiro's lemma is purely formal and the only input is the classical Shapiro's lemma and the Ind-algebraicity of $\mathcal{X}_{{}^LG}$ (which is established in Section 5). Similar arguments show that the formation of $\mathcal{X}_{{}^LG}$ is independent of the splitting field E we choose. Indeed, $\mathcal{X}_{{}^LG}$ is unique in the following sense: if $\mathcal{X}'$ is another moduli stack satisfying Theorem 2 and there exists a morphism $f : \mathcal{X}' \to \mathcal{X}_{{}^LG}$ such that (1) $f(A)$ is essentially bijective for all Artinian $W(\bar{\mathbb{F}}_p)$-algebras and (2) f_{red} is of finite type, then $\mathcal{X}' \cong \mathcal{X}_{{}^LG}$. In particular, if G is a torus, then $\mathcal{X}_{{}^LG}$ is isomorphic to the moduli of continuous Weil representations.

Concretely, the moduli stack $\mathcal{X}_{{}^LG}$ parameterizes tuples $(F, \phi_F, \gamma_F, \alpha, c)$ where (F, ϕ_F, γ_F) is a cyclotomic étale (φ, Γ)-module with ${}^LG$-structure for the p-adic field E that admits cyclotomic Kisin lattices in a weak sense, α is a tame descent datum with respect to E/K, and c is a $\widehat{G}$-level structure (see Subsection 1.2). As in the work of Pappas-Rapoport ([PR09]) and Emerton-Gee ([EG23]), the Kisin resolution plays a pivotal role in the construction of $\mathcal{X}_{{}^LG}$. When G is a ramified group, the relevant cyclotomic (φ, Γ)-modules do not admit Kisin lattices. Instead of finding a suitable replacement of Kisin lattices, we opt for the simpler method that reduces to cases where Kisin lattices do exist.

The formation of $\mathcal{X}_{{}^LG}$ is functorial in the group G and in the field K. More precisely, if ${}^LG \to {}^LH$ is an L-homomorphism, then there is a canonical change-of-group morphism

$$\mathcal{X}_{{}^LG} \to \mathcal{X}_{{}^LH}.$$

If L/K is an arbitrary extension, then there is a canonical change-of-field morphism (7.3.9)

$$\mathcal{X}_{{}^LG} \to \mathcal{X}_{L, {}^LG}$$

corresponding to the restriction map $H^1_{\mathrm{cts}}(\mathrm{Gal}_K, \widehat{G}) \to H^1_{\mathrm{cts}}(\mathrm{Gal}_L, \widehat{G})$.

1.1. The moduli stack of φ-modules with $\widehat{G}$-structure

The first step in constructing the Emerton-Gee stacks is to build the moduli stack of φ-modules. We will consider the split group case, and assume $\widehat{G} = {}^LG$. Let Λ be a finite flat $\mathbb{Z}/p^a$-algebra and let $\varphi : \Lambda((u)) \to \Lambda((u))$ be a deformation of the q-power Frobenius (i.e. $\varphi(u) \in u^q + p\Lambda((u))$).

The way representability theorems are proved in [EG21] and this paper can be summarized as follows. Write $\mathcal{R}$ for the moduli stack of $\Lambda((u))$-modules M equipped with a φ-structure $M \cong \varphi^* M$. We first solve the deformation problem and show that (after truncation) all finite type points of $\mathcal{R}$ admit a Noetherian versal ring $\operatorname{Spf} R$. Next, we show the versal morphism $\operatorname{Spf} R \to \mathcal{R}$ descends (or admits an algebraization) to $\operatorname{Spec} R \to \mathcal{R}$. Finally, Artin approximations of $\operatorname{Spec} R$ furnish smooth local charts of $\mathcal{R}$, and these smooth local charts glue into an Artin stack.

When E is ramified over $\mathbb{Q}_p$, we will encounter Frobenius endomorphisms φ such that $\varphi(\Lambda[[u]]) \not\subset \Lambda[[u]]$. In [PR09] and [EG21], $\varphi(\Lambda[[u]]) \subset \Lambda[[u]]$ is a running assumption throughout the paper, and in [EG23], it is shown that the induction construction is relatively representable and the general case is reduced to the $\varphi(\Lambda[[u]]) \subset \Lambda[[u]]$-case. The Tannakian formalism plays a crucial role when working with general groups. However, due to the non-monoidal nature of the induction construction, a different approach is required. As a result, we have to undertake the arduous task of reapplying the [PR09] construction in a more generalized setting.

We obtained general representability results in Theorem 3.7.3, which imply, for example, the following:

Example 1. *Let $\varphi : \mathbb{Z}/p^2((u)) \to \mathbb{Z}/p^2((u))$ be the endomorphism defined by $u \mapsto u^p + \frac{p}{u^r}$ such that $p + r \notin p^{\mathbb{Z}}$. The moduli stack of étale φ-modules over $\mathbb{Z}/p^2((u))$ is an Ind-algebraic stack over $\mathbb{Z}/p^2$.*

In this paper, the deformed Frobenius endomorphisms we work with are the φ-endomorphisms on the period rings $\mathbb{A}_E$ for cyclotomic (φ, Γ)-modules for the p-adic field E.

The following two observations are key to tackling the representability problem in the generalized setting: (I) the usual filtration on the loop group $\mathrm{L}\,\widehat{G}_{\mathbb{Z}/p^a}$ can be modified into a Frobenius-stable filtration (see Definition 3.2.2), and (II) by Krasner's lemma, we can ensure $\mathbb{A}^+_{\mathbb{Q}_p} \subset \mathbb{A}^+_E$ if E is tamely ramified over $\mathbb{Q}_p$ (after possibly replacing E by an unramified extension). By making observation (I), we are able to prove the Ind-representability of the moduli of Kisin lattices. Similarly, observation (II) enables us to establish that the algebraization of an étale φ-module still retains the property of étaleness. The moduli of Kisin lattices serves various technical functions; it is utilized for defining truncations on the moduli stack of étale φ-modules and is crucial in the process of algebraizing versal rings.

We solve the deformation problem by choosing an embedding $\widehat{G} \to \mathrm{GL}_N$ and show that the embedding induces a relatively representable morphism of deformation problems by checking Schlessinger's criterion. The GL_N-case is dealt with in [EG21]. The algebraization process is done in a very general setting and is monoidally functorial, with the caveat that the algebraized φ-module is not automatically étale. We proceed by using the affine morphism $\operatorname{Spec} \mathbb{A}_E \to \operatorname{Spec} \mathbb{A}_{\mathbb{Q}_p}$ to ascend the étaleness. We remark that as long as there exists a subring $\mathbb{A}_F \subset \mathbb{A}_E$ such that $\varphi(\mathbb{A}^+_F) \subset \mathbb{A}^+_F \subset \mathbb{A}^+_E$, Theorem 2 holds; we make the obvious choice $F = \mathbb{Q}_p$ when E is tame over $\mathbb{Q}_p$. The last step has been formalized in [EG21] and we only need to cite their main theorem.

1.2. The $\widehat{G}$-level structure

Recall that two L-parameters are equivalent if they are $\widehat{G}$-conjugate to each other.

In the category of (φ, Γ)-modules with ${}^L G$-structure, morphisms correspond to ${}^L G$-conjugates in the category of L-parameters. It is crucial to choose $\widehat{G}$-level structures to reduce the set of morphisms.

Let F be a (φ, Γ)-module with ${}^L G$-structure over $\mathbb{A}_{K,A}$ that corresponds to an A-family of Galois representations. We can regard $\operatorname{Gal}(E/K)$ as the L-group of the trivial group $\{*\}$. The L-homomorphism

${}^L G \to {}^L\{*\}$ induces an $\mathbb{A}_{K,A}$-morphism

$$F \to F \times^{{}^L G} {}^L\{*\} =: \bar{F}.$$

There exists an ${}^L\{*\}$-equivariant morphism $c : \bar{F} \cong \operatorname{Spec} \mathbb{A}_{E,A}$ over $\operatorname{Spec} \mathbb{A}_{K,A}$. The choice of such a c is the $\widehat{G}$-level structure we need. Note that the composition $F \to \bar{F} \cong \operatorname{Spec} \mathbb{A}_{E,A}$ endows F with a structure of (φ, Γ)-module with $\widehat{G}$-structure over $\mathbb{A}_{E,A}$.

1.3. The reduced moduli stack $\mathcal{X}_{{}^L G,\mathrm{red}}$

We begin our study of $\mathcal{X}_{{}^L G}$ by examining its reduction $\mathcal{X}_{{}^L G,\mathrm{red}}$, with the key input being the Langlands-Shelstad factorization theorem established in [L23]. The theorem establishes a dichotomy for a mod p L-parameter $\mathrm{Gal}_K \to {}^L G(\bar{\mathbb{F}}_p)$, which states that it is either parabolic or elliptic. The former case corresponds to a situation where the L-parameter factors through ${}^L P$ for some proper maximal parabolic P, while the latter case corresponds to a situation where the L-parameter factors through ${}^L S$ for some maximally unramified elliptic torus S.

One can obtain a good understanding of $\mathcal{X}_{{}^L G,\mathrm{red}}$, which is ultimately shown to be a finite type reduced algebraic stack over $\bar{\mathbb{F}}_p$, by analyzing the images of $\mathcal{X}_{{}^L P,\mathrm{red}}$ and $\mathcal{X}_{{}^L S,\mathrm{red}}$ in $\mathcal{X}_{{}^L G,\mathrm{red}}$. We have the following structural results:

- The moduli stack $\mathcal{X}_{{}^L S}$ can be identified with the moduli stack of continuous representations of the Weil group W_K.
- Write ${}^L L$ for the Levi factor of ${}^L P$. The moduli stack $\mathcal{X}_{{}^L L}$ admits a constructible stratification $\{\mathcal{Z}_i\}$ such that the coarse moduli space $\mathcal{V}_i$ of $\mathcal{X}_{{}^L P} \times_{\mathcal{X}_{{}^L L}} \mathcal{Z}_i$ has a concrete description. There exists a tower $\mathcal{V}_i = \mathcal{V}_{i,0} \to \mathcal{V}_{i,1} \to \cdots \to \mathcal{V}_{i,m} = \mathcal{Z}_i$ such that each $\mathcal{V}_{i,k}$ is a closed subscheme of an affine bundle $\mathcal{E}_{i,k}$ over $\mathcal{V}_{i,k+1}$.

If ${}^L G = \mathrm{GL}_n$, then $\mathcal{V}_i = \mathcal{V}_{i,m}$ is a vector bundle over $\mathcal{Z}_i$. In general, $\mathcal{V}_{i,k}$ is the vanishing locus of *higher cup products* for Herr complexes in $\mathcal{E}_{i,k}$. Although $\mathcal{V}_{i,k}$ and $\mathcal{E}_{i,k}$ are not identical in general, we can ignore this discrepancy to establish both an upper and a lower bound for the dimension of $\mathcal{X}_{{}^L G,\mathrm{red}}$ in terms of the dimension of the various $\mathcal{X}_{{}^L L,\mathrm{red}}$. In the case where ${}^L G = \mathrm{GL}_n$, the upper and lower bounds coincide. However, in general, understanding the discrepancy is crucial, and we have developed the so-called "Lyndon-Demuškin method" in [L21] to comprehend the vanishing locus of cup products for classical groups. We will not touch upon dimension computation in this paper; it will be addressed on a case-by-case basis in a subsequent paper.

1.4. Herr complexes

Herr complexes are algebraic interpolations of the Galois cohomology. In Section 8, we establish the (infinitesimal) obstruction theory for the stack $\mathcal{X}_{{}^L G}$, which implies the Noetherian formal algebraicity of $\mathcal{X}_{{}^L G}$. Under certain technical hypothesis, we show that the standard Galois cohomology identity

$$H^\bullet_{\mathrm{cont}}(\mathrm{Gal}_E, M)^{\mathrm{Gal}(E/K)} = H^\bullet_{\mathrm{cont}}(\mathrm{Gal}_K, M)$$

can be generalized to algebraic families of L-parameters.

In Section 9, we study a different sort of obstruction theory. We define higher cup products for Herr complexes and use them to study non-abelian unipotent extensions of étale (φ, Γ)-modules. It is worth noting that for ${}^L G = \mathrm{GL}_n$, the unipotent radical of maximal proper parabolics of ${}^L G$ is abelian. As a result, the study of cup products is unnecessary. For classical groups such as unitary group U_n, symplectic group GSp_{2n}, or orthogonal group GO_n, it is sufficient to analyze the usual cup products. However, for exceptional groups, the investigation of higher cup products is necessary.

1.5. Langlands functoriality for Serre weights

Under the assumption that the reductive quotient of a superspecial parahoric of G has simply connected derived subgroup, we expect that the irreducible components of $\mathcal{X}_{{}^LG,\mathrm{red}}$ are labeled by Serre weights for G (see [L23]). The study of the irreducible components of $\mathcal{X}_{{}^LG,\mathrm{red}}$ requires the calculation of the dimension of $\mathcal{X}_{{}^LG,\mathrm{red}}$, which we have already explained in Subsection 1.3.

An application of the geometric interpretation of Serre weights enables us to study Langlands functoriality for Serre weights. Given an L-homomorphism ${}^LG \to {}^LH$, we have a canonical morphism $\mathcal{X}_{{}^LG} \to \mathcal{X}_{{}^LG}$. By analyzing the relation of top-dimensional cycles in $\mathcal{X}_{{}^LG,\mathrm{red}}$ and $\mathcal{X}_{{}^LH,\mathrm{red}}$, we can predict the transfer maps for Serre weights.

1.6. Notations and conventions

${}^f\mathrm{Rep}_H$	the category of representations of H on finite projective $\mathbb{Z}_p$-modules
$i : {}^LG \hookrightarrow \mathrm{GL}_N$	a fixed faithful (algebraic) representation

We will need to following lemma.

1.7. Lemma For each $\mathbb{Z}_p$-scheme X, the category of ${}^LG$-torsors on X is equivalent to the category of fiber functors from ${}^f\mathrm{Rep}_{{}^LG}$ to Vect_X.

Proof. It is [Lev13, Theorem 2.5.2]. Note that the connectedness assumption is only used in [Lev13, Proposition C.1.8]. Since ${}^LG$ as a scheme is a disjoint union of $\widehat{G}$, [Lev13, Proposition C.1.8] holds for ${}^LG$. $\square$

Since the Langlands dual group ${}^LG$ is independent of the choice of inner form, we assume throughout the paper that G is a quasi-split group.

All stacks will be stacks in the fppf topology. An algebraic stack is by definition an Artin stack [Stacks, Tag 026N].

1.7.1. A list of stacks

Stack	Description	Definition
$\mathcal{C}_{\widehat{G}}$	moduli of Kisin lattices	Definition 3.1.5
$\mathcal{Z}_{\widehat{G}}$	moduli of étale φ-modules	Definition 3.1.5
$\mathcal{R}_{\widehat{G}}$	moduli of admissible étale φ-modules	Definition 3.1.5
$\mathcal{Z}_{E,{}^LG}$	moduli of étale φ-modules over $\mathbb{A}_E$	Definition 5.3.1
$\mathcal{R}_{E,{}^LG}$	moduli of rigidifiable étale φ-modules over $\mathbb{A}_E$	Definition 5.3.1
$\mathcal{Z}_{K,{}^LG}$	moduli of étale φ-modules over $\mathbb{A}_K$	Definition 5.2.6
$\mathcal{R}_{K,{}^LG}$	moduli of E-admissible étale φ-modules over $\mathbb{A}_E$	Definition 5.2.6
$\mathcal{Z}^{\gamma}_{{}^LG}$	moduli of étale (φ,γ)-modules	Definition 6.1.1
$\mathcal{Z}^{\Gamma}_{{}^LG}$	moduli of étale (φ,Γ)-modules	Definition 6.1.1
$\mathcal{R}^{\gamma}_{{}^LG}$	moduli of E-admissible étale (φ,γ)-modules	Definition 6.1.1
$\mathcal{R}^{\Gamma}_{{}^LG}$	moduli of E-admissible étale (φ,Γ)-modules	Definition 6.1.1
$\mathcal{X}_{{}^LG}$	moduli of L-parameters	Definition 7.1.1
$\mathcal{X}_{L,{}^LG}$	moduli of (φ,G_L)-modules corresponding to L-parameters	Definition 7.3.8
$\mathcal{R}_{L,{}^LH}$	moduli of (φ,G_L)-modules	Definition 7.3.8
$\mathcal{R}_{L,{}^LH,\widehat{H}}$	moduli of (φ,G_L)-modules with $\widehat{G}$-level structure	Definition 7.3.8

2. Some commutative algebra

2.1. Elementary facts about pushforward and pullback

Let R be a Noetherian ring complete with respect to an ideal $I \subset R$.

Consider the following diagram

$$
\begin{tikzcd}
\mathrm{Spf}\, \widehat{R((u))} \arrow[dr, "\mathfrak{c}"] & & \\
 & \mathrm{Spec}\, R((u)) \arrow[r, "j"] & \mathrm{Spec}\, R[[u]] \\
\mathrm{Spec}\, \widehat{R((u))} \arrow[ur, "c"] & & \\
\mathrm{Spec}\, R/I^n((u)) \arrow[u, "i_n"] & \mathrm{Spec}\, R/I^n((u)) \arrow[uu, "i_n"] & \mathrm{Spec}\, R/I^n[[u]] \arrow[uu, "i_n"]
\end{tikzcd}
$$

where $\widehat{R((u))}$ is the I-adic completion of $R((u))$ and the morphisms c, j, and i_n are induced from the corresponding ring maps.

We collect the following simple facts. Note that c^*, j^*, i_n^* are simply base change; c_*, j_* and $(i_n)_*$ are forgetful functors that preserve the underlying abelian groups; and $\mathfrak{c}^*$ is by definition the I-adic completion functor.

2.1.1. Fact (1) We have the following identities.

(a) $j^*j_* = \mathrm{id} : \mathrm{QCoh}(R((u))) \to \mathrm{QCoh}(R((u)))$;
(b) $i_n^*(i_n)_* = \mathrm{id} : \mathrm{QCoh}(R/I^n((u))) \to \mathrm{QCoh}(R/I^n((u)))$;
(c) $i_n^*(i_n)_* = \mathrm{id} : \mathrm{QCoh}(R/I^n[[u]]) \to \mathrm{QCoh}(R/I^n[[u]])$;
(d) $i^*i_* = \mathrm{id} : \mathrm{QCoh}(\widehat{R((u))}) \to \mathrm{QCoh}(\widehat{R((u))})$;
(e) $\mathfrak{c}^*c_* = \mathrm{id} : \mathrm{Coh}(\widehat{R((u))}) \to \mathrm{Coh}(\widehat{R((u))})$;
(f) $c^* = \mathfrak{c}^* : \mathrm{Coh}(R((u))) \to \mathrm{Coh}(\widehat{R((u))})$.

(2) If $M \in \mathrm{Coh}(R((u)))$ and $N \in \mathrm{QCoh}(\widehat{R((u))})$ is flat, then $M \subset c_*N$ implies $c^*M \subset N$ (the map exists by adjunction).

Proof. (1-a,b,c,d) Clear.

(1-e) By [Stacks, Tag 00MA], a finite $\widehat{R((u))}$-module is automatically I-adically complete.

(1-f) It is part (3) of [Stacks, Tag 00MA].

(2) By [Stacks, Tag 0315], $\mathfrak{c}^*M \subset \mathfrak{c}^*c_*N$. By part (1-e), $\mathfrak{c}^*c_*N = N$. By part (1-f), $\mathfrak{c}^*M = c^*M$. □

2.2. Mittag-Leffler systems of modules

See [Stacks, Tag 0594] for the basic definitions of Mittag-Leffler systems.

2.2.1. Definition Let (A_i, φ_{ij}), $(B_i, \varphi_{i,j})$ be Mittag-Leffler directed inverse systems of R-modules. We say

- $(A_i, \varphi_{ij}) \subset (B_i, \varphi_{ij})$ if $A_i \subset B_i$ for each i;
- $(A_i, \varphi_{ij}) \models (B_i, \varphi_{ij})$ if $(A_i, \varphi_{ij}) \subset (B_i, \varphi_{ij})$ and $\varphi_{ji}(A_j) = \varphi_{ji}(B_j)$ for all i and all $j \gg i$.

2.2.2. Lemma Let (A_i, φ_{ij}), (B_i, φ_{ij}) be Mittag-Leffler directed inverse systems of R-modules, and let (D_i, φ_{ij}) be a system that contains both (A_i, φ_{ij}), (B_i, φ_{ij}). Then $(A_i \cap B_i, \varphi_{ij})$ is Mittag-Leffler.

Moreover if $(C_i, \varphi_{ij}) \subset (D_i, \varphi_{ij})$ and $(B_i, \varphi_{ij}) \models (C_i, \varphi_{ij})$, then $(A_i \cap B_i, \varphi_{ij}) \models (A_i \cap C_i, \varphi_{ij})$.

Proof. Clear. □

2.2.3. Lemma For each $t \in R$, if (A_i, φ_{ij}) is a Mittag-Leffler directed inverse system of R-modules then so is $(A_i \otimes_R R[1/t], \varphi_{ij})$.

Moreover if $(A_i, \varphi_{ij}) \models (B_i, \varphi_{ij})$, then $(A_i \otimes_R R[1/t], \varphi_{ij}) \models (B_i \otimes_R R[1/t], \varphi_{ij})$.

Proof. It suffices to show that if $f : M \to N$ is a homomorphism of R-modules, then $\mathrm{Im}(f) \otimes_R R[1/t] = \mathrm{Im}(f \otimes_R R[1/t])$. Note that $M \to f(M)$ is surjective and $f(M) \subset N$ is injective. After localization, $M[1/t] \to f(M)[1/t]$ is surjective and $f(M)[1/t] \to N[1/t]$ is injective since $R[1/t]$ is R-flat. So $f(M)[1/t]$ is the image of $f[1/t]$. □

2.2.4. Lemma Fix an integer $c > 0$. Let (A_i, φ_{ij}), (B_i, φ_{ij}) be Mittag-Leffler directed inverse systems of R-modules where indexes $i \in \mathbb{Z}$. Assume there exists surjective homomorphisms $f_i : A_i \to B_i$, $g_i : B_i \to A_{i-c}$ for all i such that

- Both f_i, g_i are compatible with transition maps φ_{ij};
- $g_i \circ f_i = \varphi_{i,i-c}$.

Then $\varprojlim_i A_i = \varprojlim_i B_i$.

Proof. Clear since both $\lim f_i \circ \lim g_i$ and $\lim g_i \circ \lim f_i$ are identities. □

2.3. Non-flat descent of vector bundles

2.3.1. Definition Given a tuple (U, X, Y, j, π) where $j : U \to X$ is a scheme-theoretically dominant flat morphism of affine schemes and $\pi : Y \to X$ is a scheme-theoretically dominant proper morphism of schemes. Denote by $\mathrm{Vect}_{U,X,Y}$ the category of triples (V, K, θ) where V is a locally free coherent sheaf over U, K is a locally free coherent sheaf over Y, and θ is an isomorphism $\theta : \pi^* V \cong j^* K$. For ease of notation, we will drop θ and write $(V, K, \theta) = (V, K)$ and $\pi^* V = j^* K$.

Write $Y_U := U \times_X Y$. We have a diagram

$$\begin{tikzcd} Y_U \arrow[r, "j"] \arrow[d, "\pi"] & Y \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where all morphisms are scheme-theoretically dominant.

2.3.2. Lemma We have $\pi_* j_* \pi^* = \pi_* \pi^* j_*$ as functors from coherent sheaves over U to quasi-coherent sheaves over X.

Proof. Since the maps involved are either proper or affine, pushforward preserve quasi-coherence; since X is affine, to show $\pi_* j_* \pi^* F = \pi_* \pi^* j_* F$ it suffices to check the global sections. Note that by the pullback-pushforward adjunction, we have a canonical map $\pi^* j_* \to j_* \pi^*$. Since $j^* \pi_* j_* \pi^* F$ is in set-theoretic bijection with $\pi_* j_* \pi^* F$, it suffices to show $j^* \pi_* j_* \pi^* F = j^* \pi_* \pi^* j_* F$ which is clear since $j^* j_* = 1$ and $j^* \pi_* = \pi_* j^*$ by flat base change. □

By the previous lemma, for each $(V,K) \in \mathrm{Vect}_{U,X,Y}$, $\pi_*\pi^*j_*V = j_*j^*\pi_*K$. Since π and j are both scheme-theoretically dominant, the units of the pullback-pushforward adjunctions $1 \to \pi_*\pi^*$ and $1 \to j_*j^*$ are inclusion of quasi-coherent sheaves.

2.3.3. Definition Define a functor $(j,\pi)_* : \mathrm{Vect}_{U,X,Y} \to \mathrm{QCoh}_X$ by $(V,K) \mapsto j_*V \cap \pi_*K$ $(\subset \pi_*\pi^*j_*V)$.

2.3.4. Lemma $(j,\pi)_* : \mathrm{Vect}_{U,X,Y} \to \mathrm{QCoh}_X$ is a lax monoidal functor (see Appendix B for definitions).

Proof. Let (V_1,K_1), $(V_2,K_2) \in \mathrm{Vect}_{U,X,Y}$. Since both j_* and π_* are lax monoidal functors, we have maps

$$(j_*V_1) \otimes (j_*V_2) \to j_*(V_1 \otimes V_1) \subset \pi_*\pi^*j_*(V_1 \otimes V_2)$$
$$(\pi_*K_1) \otimes (\pi_*K_2) \to \pi_*(K_1 \otimes K_1) \subset \pi_*\pi^*j_*(V_1 \otimes V_2)$$

whose restriction to $(j,\pi)_*(V_1,K_1) \otimes (j,\pi)_*(V_2,K_2)$ coincide. The restriction defines a lax monoidal structure on $(j,\pi)_*$. □

2.3.5. Lemma If j is an open immersion, $j^*(j,\pi)_*(V,K) = V$.

Proof. We have $j_*j^*(j_*V \cap \pi_*K) = j_*V \cap j_*j^*\pi_*K = j_*V \cap j_*\pi_*j^*K = j_*V \cap \pi_*\pi^*j_*V = j_*V$. Note that $j^*j_* = \mathrm{id}$. □

2.3.6. Definition (Algebraic disks) Let R be a Noetherian ring. Let $Y \to \operatorname{Spec} R$ be a proper morphism. By the Grothendieck algebraization theorem [Stacks, Tag 089A], there exists a proper scheme morphism $Y' \to \operatorname{Spec} R[[u]]$ such that for each integer n, $Y' \otimes_{R[[u]]} R[[u]]/u^n \to \operatorname{Spec} R[[u]]/u^n$ is the base change of $Y \to \operatorname{Spec} R$ to $\operatorname{Spec} R[u]/u^n$. Write $Y[[u]]$ for Y'.

2.3.7. Proposition Let $R = \varprojlim_n R/I^n$ be a Noetherian complete ring. Let $\pi : X \to \operatorname{Spec} R$ be a scheme-theoretically dominant, proper morphism. There exists a lax monoidal functor

$$\xi : \mathrm{Vect}_{\operatorname{Spec} \widehat{R((u))}, \operatorname{Spec} R[[u]], X[[u]]} \to \mathrm{Coh}_{\operatorname{Spec} R((u))}$$

which becomes the forgetful functor after composing with the base change functor $\mathrm{Coh}_{\operatorname{Spec} R((u))} \to \mathrm{Coh}_{\operatorname{Spf} \widehat{R((u))}}$.

Proof. Write R_n for R/I^n, and write X_n for $X \times_{\operatorname{Spec} R} \operatorname{Spec} R_n$. The arrows in the following diagram

$$\begin{tikzcd} X_n((u)) \arrow[r, "j"] \arrow[d, "\pi"] & X_n[[u]] \arrow[d, "\pi"] \\ \operatorname{Spec} R_n((u)) \arrow[r, "j"] & \operatorname{Spec} R_n[[u]] \end{tikzcd}$$

are all scheme-theoretically dominant. Write $i_n : \operatorname{Spec} R_n \to \operatorname{Spec} R$ for the embedding. Write j for $\operatorname{Spec} R((u)) \to \operatorname{Spec} R[[u]]$. For $(V,K) \in \mathrm{Vect}_{\operatorname{Spec} \widehat{R((u))}, \operatorname{Spec} R[[u]], \mathfrak{X}[[u]]}$, set

$$\kappa_n(V,K) := (j,\pi)_* i_n^*(V,K)$$

and $\kappa := \varprojlim_n \kappa_n$. Define $\xi(V,K) := j^*\kappa(V,K)$.

By the theorem on formal functions, $\varprojlim_n \pi_* i_n^* K = \pi_* K$ is coherent and thus the submodule $\kappa(V,K)$ is also coherent. So $\xi(V,K) \in \mathrm{Coh}(R((u)))$.

It is clear that ξ is a lax monoidal functor. So it remains to show ξ composed with the I-adic completion functor is the forgetful functor; in other words,

$$(\dagger) \quad \varprojlim_n i_n^* \xi(V,K) = \varprojlim_n j^* i_n^* \kappa(V,K) = V.$$

By the Artin-Rees lemma ([Stacks, Tag 00IN]), there exists a constant number $c > 0$ such that

$$(I^n \pi_* K) \cap \kappa(V,K) = I^{n-c}(I^c \pi_* K \cap \kappa(V,K))$$

for all $n \geq c$. So we have

$$I^n \kappa(V,K) \subset (I^n \pi_* K) \cap \kappa(V,K)$$
$$(I^{n+c} \pi_* K) \cap \kappa(V,K) \subset I^n \kappa(V,K)$$

for all n. Note that

$$\frac{\kappa(V,K)}{(I^n \pi_* K) \cap \kappa(V,K)} = \mathrm{Im}(\kappa(V,K) \to i_n^* \pi_* K) =: A_n.$$

The following maps

$$A_n \to i_n^* \kappa(V,K)$$
$$i_n^* \kappa(V,K) \to A_{n+c}$$

are surjective for all n. It follows that

$$\varprojlim_n j^* i_n^* \kappa(V,K) = \varprojlim_n j^* A_n.$$

By [Stacks, Tag 02OB], there exists a positive integer c' such that if we write $K_n := \mathrm{Im}(\pi_* K \to \pi_* i_n^* K)$, then $K_n \to i_{n-c'}^* \pi_* K$ is surjective for all $n \geq c'$. Write $B_n := \mathrm{Im}(\kappa(V,K) \to \pi_* i_n^* K) \subset K_n$. We have surjective maps $A_n \to B_{n-c'}$ for all $n \geq c'$. Note that $(B_n) \models (\kappa_n(V,K))$. So

$$\varprojlim_n j^* \kappa_n(V,K) = \varprojlim_n j^* B_n.$$

By Lemma 2.3.5,

$$\varprojlim_n j^* \kappa_n(V,K) = \varprojlim_n i_n^* V = V.$$

Note that both (A_n) and (B_n) are Mittag-Leffler inverse systems with surjective transition maps, and that $\varprojlim_n A_n = \varprojlim_n B_n = \kappa(V,K)$; it is possible to choose an infinite sequence of increasing integers $m_1 < m_2 < \dots$ such that A_{m_i} is a quotient of $B_{m_{i+1}-c'}$ for all i. Now it is clear that

$$\varprojlim_n j^* A_n = \varprojlim_i j^* A_{m_i} = \varprojlim_i j^* B_{m_i - c'} = \varprojlim_i j^* B_n = V.$$

So we are done. $\square$

2.3.8. Affine pushforward Let $R[[v]] \subset R[[u]]$ be a subring such that $R[[u]]$ is a finite projective $R[[v]]$-module. Write $\zeta : \operatorname{Spec} R[[u]] \to \operatorname{Spec} R[[v]]$ for the corresponding morphism. Also denote by ζ the morphism $X[[u]] \to X[[v]]$. Note that $\zeta_*(j,\pi)_* = (j,\pi)_*\zeta_*$ since $\zeta_* j_* = j_*\zeta_*$ and $\zeta_*\pi_* = \pi_*\zeta_*$. Since ζ is an affine morphism, ζ_* commutes with i_n^* (see the proof of Proposition 2.3.7 for the notation), and $\zeta_*\kappa_n(V,K) = \kappa_n\zeta_*(V,K)$. So we conclude that $\zeta_* \circ \xi = \xi \circ \zeta_*$ where ξ is the functor defined in Proposition 2.3.7. Note that $\xi \circ \zeta_*(V,K)$ is in set-theoretical bijection with $\xi(V,K)$.

2.4. Local contractions

2.4.1. Definition An endomorphism $\varphi : A((u)) \to A((u))$ is said to be *locally contracting* if there exists an integer N and a real number $\lambda > 1$ such that for all integers $n > N$, $\varphi(u^n) \in u^{\lfloor \lambda n \rfloor} A[[u]]$. We call $\lambda(\varphi) := \lambda$ the contracting factor of φ, and call $d_\lambda(\varphi) := \max\limits_{N}(\frac{1}{N})$ the contracting diameter of φ (of contracting factor λ).

For any ring R, write $M_d(R)$ for $d \times d$-matrix with R-coefficients. Set $U_n(A) := 1 + u^n M_d(A[[u]])$ and $V_n(R) = \{x \in \mathrm{GL}_d(A((u)))|x, x^{-1} \in u^{-n}M_d(A[[u]])\}$.

2.4.2. Lemma Assume $\varphi : A((u)) \to A((u))$ is locally contracting of contracting factor λ. Then for all $n > \max(\frac{2}{\lambda-1}m, \frac{1}{d_\lambda(\varphi)})$,

(1) For each $g \in U_n(A)$, $x \in V_m(A)$, there exists a unique $h \in U_n(A)$ such that $g^{-1}x\varphi(g) = h^{-1}x$.
(2) For each $h \in U_n(A)$, $x \in V_m(A)$, there exists a unique $g \in U_n(A)$ such that $g^{-1}x\varphi(g) = h^{-1}x$.

Proof. We follow the proof of [PR09, Prop. 2.2] and [EG23, Lemma 5.2.9].

(1) Since $h^{-1} = g^{-1}x\varphi(g)x^{-1}$, the uniqueness of h is clear. Write $g^{-1} = 1 + u^n X$ with $X \in M_d(A[[u]])$, and $\varphi(g) = 1 + u^s Y$ with $Y \in M_d(A[[u]])$. We can take $s \geq \lambda n$ since φ is locally contracting of contracting factor λ. Then $g^{-1}x\varphi(g)x^{-1} = (1+u^x X)(1+u^{s-2m}u^m xYu^m x^{-1}) \in U_n(A)$ since $s - 2m \geq \lambda n - 2m > n + (\lambda-1)n - 2m \geq n$.

(2) We first show the uniqueness of g, for which it is enough to show $g^{-1}x\varphi(g) = x$ implies $g = 1$. Write $g = 1 + u^s X$, $X \in M_d(A[[u]])$. We have $u^s X = \varphi(u^s)x\varphi(X)x^{-1}$. If $s > \frac{2}{1-\lambda}m$, $u^{s+1}|\varphi(u^s)u^{-2m}$. So $X \in uM_d(A[[u]])$.

Finally we show the existence of g. Let $x' := h^{-1}x$. Set $x_0 = x$, $h_0 = h$. We inductively define sequences (h_i), (x_i) by setting $x_i = h_{i-1}^{-1}x_{i-1}\varphi(h_{i-1})$, $h_i = (x')^{-1}x_i$. The argument in the last paragraph shows that $h_i \in U_{n+i}$ for all i. Set $g_i = h_0 h_1 \dots h_i$, then g_i tends to some limit $g \in U_n$ and since for all i, $g_i^{-1}x\varphi(g_{i-1}) = h^{-1}x$, in the limit $g^{-1}x\varphi(g) = h^{-1}x$. □

2.5. Frobenius contractions

We first show that any p-adic deformation of the p-power Frobenius map is locally contracting mod p^a.

2.5.1. Lemma Let $\varphi : A((u)) \to A((u))$ be an endomorphism such that $\varphi(u) \in u^q + pA((u))$. If there exists an integer $a > 0$ such that $p^a A = 0$, then φ is locally contracting of contracting factor λ for any $1 < \lambda < q$.

Proof. Write $\varphi(u) = u^q + pY$, $Y \in A((u))$. There exists an integer $N \geq 0$ such that $u^N Y \in A[[u]]$. We have

$$\varphi(u^{M+a-1}) = (u^q + pY)^{M+a-1} = \sum_{i=0}^{a-1} \binom{M+a-1}{i} u^{q(M+a-1-i)}(pY)^i.$$

whose lowest degree term has degree at least $qM-(a-1)N$. Note that both a and N are constant numbers, and we have $\lim_{M\to\infty}\frac{qM-(a-1)N}{M+a-1}=q$. $\square$

2.5.2. Definition An endomorphism $\varphi:\Lambda((u))\to\Lambda((u))$ is said to be a *Frobenius contraction* if there exists an integer N such that

- $\varphi^{(N)}(u)=\varphi(\varphi(\cdots\varphi(u)))\in u^q+p\Lambda((u))$,
- $\varphi^{(N)}$ is Λ-linear.

An étale φ-module over $\Lambda((u))$ is a finitely generated $\Lambda((u))$-module M, equipped with an $\Lambda((u))$-isomorphism

$$\phi_M:\varphi^*M:=\Lambda((u))\otimes_{\varphi,\Lambda((u))}M\to M.$$

We will need the following descent theorem which strengthens [EG21, Theorem 5.5.20].

2.5.3. Lemma Let R be a complete Noetherian local $\mathbb{F}_p$-algebra with maximal ideal $\mathfrak{m}$, and let $\varphi:R((u))\to R((u))$ be the R-linear map that sends u to u^q.

Let $M_1\to M_2\to M_3$ be a sequence of finite free étale φ-modules over $R((u))$. If $0\to\widehat{M_1}\to\widehat{M_2}\to\widehat{M_3}\to 0$ is a short exact sequence (where $(-)^\wedge$ denotes the $\mathfrak{m}$-adic completion), then $0\to M_1\to M_2\to M_3\to 0$ is also an exact sequence.

Proof. Let S be a ring which contains an element u. Denote by $S\langle-\rangle$ the u-adic completion of $S[-]$.

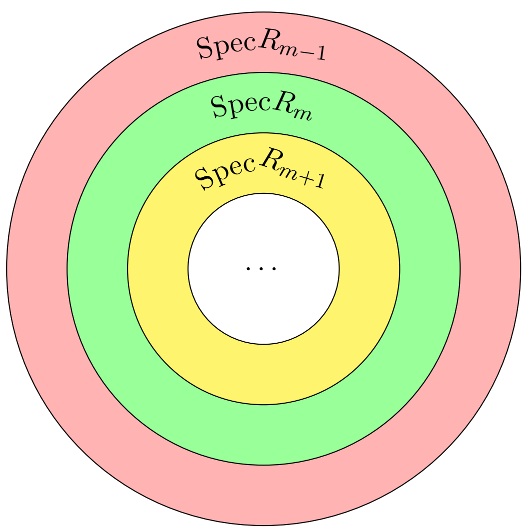
By the Cohen structure theorem, $R=K[[T_1,\dots,T_n]]/I$ for some field extension K of $\mathbb{F}_p$ and some ideal I. For each integer $m\geq 0$, we define

$$R_m:=R[u]\langle\frac{T_1}{u^{q^m}},\dots,\frac{T_n}{u^{q^m}}\rangle[1/u]$$

and

$$R_{-m}:=R[u]\langle\frac{T_1^{q^m}}{u},\dots,\frac{T_n^{q^m}}{u}\rangle[1/u].$$

It is helpful to visualize $\operatorname{Spec}R_m$ $(-\infty<m<\infty)$ as disks of various radii.



We advise the readers to read [EG21, 5.5.20] for the following facts:

(i) There are flat homomorphisms of $R((u))$-algebras $R_m \hookrightarrow R_{m+1}$ determined by $T_i \mapsto T_i$ and $R((u))$-isomorphisms
$$R((u)) \otimes_{\varphi, R((u))} R_m \xrightarrow{\sim} R_{m+1}$$
for $m = \dots, -1, 0, 1, \dots$;

(ii) Write $\varphi_m : R_m \to R_{m+1}$ for the composition $R_m \xrightarrow{x \mapsto 1 \otimes x} R((u)) \otimes_{\varphi, R((u))} R_m \xrightarrow{\sim} R_{m+1}$, which is a faithfully flat map;

(iii) The ring $R_\infty := \varinjlim_{m \geq 0} R_m$ is faithfully flat over $\widehat{R((u))}$, the $\mathfrak{m}$-adic completion of $R((u))$;

(iv) Each maximal ideal $\mathfrak{p}$ of $R((u))$ comes from a maximal ideal of R_m for some $m \leq 0$. More precisely, if $R((u)) \to L$ is a surjection onto a field L (which is necessarily a finite extension of $K((u))$), then it factors through a surjection $R_m \to L$ for some $m \leq 0$.

We will use the following constructions.

(I) For each $R((U))$-module M, write $j_m^* M$ for $R_m \otimes_{R((u))} M$, which can be interpreted geometrically as restriction to the "disk" $\operatorname{Spec} R_m$.

(II) For each R_m-module M, write φ_m^* for $R_{m+1} \otimes_{\varphi_m, R_m} M$, which can be interpreted geometrically as the Frobenius amplification of M.

The following fact is crucial:
$$\varphi_m^* j_m^* M = j_{m+1}^* \varphi^* M.$$

Since R_∞ is faithfully flat over $\widehat{R((u))}$, the sequence
$$0 \to M_1 \otimes_{R((u))} R_\infty \to M_2 \otimes_{R((u))} R_\infty \to M_3 \otimes_{R((u))} R_\infty \to 0$$
is exact. it is possible to choose a basis $\{b_1, b_2, \dots, b_s; c_1, c_2, \dots, c_t\}$ of $M_2 \otimes_{R((u))} R_\infty$ such that the image of $M_1 \otimes_{R((u))} R_\infty$ is generated by $\{b_1, \dots, b_s\}$ and each element of $M_3 \otimes_{R((u))} R_\infty$ can be lifted to an element of the submodule generated by $\{c_1, \dots, c_t\}$. Since $R_\infty = \varinjlim_m R_m$, there exists some $N \geq 0$ such that $\{b_1, b_2, \dots, b_s; c_1, c_2, \dots, c_t\} \subset M_2 \otimes_{R((u))} R_N$ and as a consequence, the sequence
$$0 \to j_N^* M_1 \to j_N^* M_2 \to j_N^* M_3 \to 0$$
is exact.

Since $\varphi : R((u)) \to R((u))$ is faithfully flat and thus the base change $\varphi_m : R_m \to R((u)) \otimes_{\varphi, R((u))} R_m \cong R_{m+1}$ is also faithfully flat. We have a commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & j_m^* M_1 \arrow[r] & j_m^* M_2 \arrow[r] & j_m^* M_3 \arrow[r] & 0 \\
0 \arrow[r] & j_m^* \varphi^* M_1 \arrow[r] \arrow[u, "\phi_{M_1}"] & j_m^* \varphi^* M_2 \arrow[r] \arrow[u, "\phi_{M_2}"] & j_m^* \varphi^* M_3 \arrow[r] \arrow[u, "\phi_{M_3}"] & 0 \\
0 \arrow[r] & \varphi_{m-1}^* j_{m-1}^* M_1 \arrow[r] \arrow[u, equal] & \varphi_{m-1}^* j_{m-1}^* M_2 \arrow[r] \arrow[u, equal] & \varphi_{m-1}^* j_{m-1}^* M_3 \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$
where all vertical maps are isomorphisms. Hence the short exactness of
$$0 \to j_m^* M_1 \to j_m^* M_2 \to j_m^* M_3 \to 0$$
implies the short exactness of
$$0 \to j_{m-1}^* M_1 \to j_{m-1}^* M_2 \to j_{m-1}^* M_3 \to 0.$$

To show $0 \to M_1 \to M_2 \to M_3$ is exact, it suffices to show that for each maximal ideal $\mathfrak{p}$ of $R((u))$, the localisation at $\mathfrak{p}$ of this sequence is exact. We have already mentioned that there exists some integer $m \leq 0$ and some maximal ideal $\mathfrak{q}$ of R_m which lies above $\mathfrak{p}$. Since $R((u)) \to R_m$ is flat, $R((u))_{\mathfrak{p}} \to (R_m)_{\mathfrak{q}}$ is faithfully flat. The proof is now complete. $\square$

The following elementary lemma allows us to generalise the above lemma to a general $\mathbb{Z}_p$-algebra A.

2.5.4. Lemma Let A be a ring. Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism such that all the closed points of $\operatorname{Spec} A$ are contained in the image of f. Let $M_1 \to M_2 \to M_3$ be a sequence of finite projective A-modules such that

$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$$

is a short exact sequence. Then

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is also a short exact sequence.

Proof. Since the property of being a short exact sequence is local, it suffices to consider the case where both A and B are local rings, and $A \to B$ is a local homomorphism. We can replace B by its residue field. So assume A is a local ring with maximal ideal I, and $B = A/I$.

Fix a spliting $\bar{\psi} : M_1/IM_1 \oplus M_3/IM_3 \xrightarrow{\sim} M_2/IM_2$ of the short exact sequence. Let $\psi : M_1 \oplus M_3 \to M_2$ be an arbitrary lift of $\bar{\psi}$ which is compatible with the given lift of the short exact sequence; since $M_2 \to M_3$ is surjective by Nakayama's lemma, such a lift ψ exists. By Nakayama's lemma again, ψ is a surjective homomorphism of finite free A-modules of the same rank, and is thus an isomorphism (by a determinant argument). $\square$

2.6. Height theory Let $\varphi : \Lambda((u)) \to \Lambda((u))$ be a ring endomorphism.

2.6.1. Definition A *height theory* for φ is an element $v \in \Lambda((u))$ such that

- (H0a) v is invertible;
- (H0b) $\Lambda((u))$ is v-adically complete;
- (H1) $\Lambda((u))$ is a finite projective $\Lambda((v))$-module;
- (H2) $\varphi(\Lambda[[v]]) \subset \Lambda[[v]]$;
- (H3) There exists an element $u' \in \Lambda((u))$ such that $\Lambda((u')) = \Lambda((u))$ and $\Lambda[[v]] \subset \Lambda[[u']]$;
- (H4) $\Lambda((v)) \otimes_{\varphi,\Lambda((v))} \Lambda((u)) \xrightarrow{x\otimes y \mapsto x\varphi(y)} \Lambda((u))$ is a bijection.

Note that (H0a) and (H0b) guarantee that the inclusion $\Lambda[v] \subset \Lambda((u))$ factors through $\Lambda((v))$.

2.6.2. Example Let $\varphi : \mathbb{Z}/p^2((u)) \to \mathbb{Z}/p^2((u))$ be the endomorphism sending u to $u^p + \frac{2p}{u^{p-2}}$. Set $v = u^2$. We have $\varphi(v) = \varphi(u)^2 = u^{2p} + 4pu^{2p-2} = v^p + 4pv^{p-1} \in \Lambda[[v]]$. Note that if $p \neq 2$, $u = v^{(p-1)/2}(\varphi(u) - 2pv\varphi(u)^{-1})$, and thus v is a height theory for φ.

2.7. Thickening

2.7.1. Lemma Let $f : X \to \operatorname{Spec} A$ be a morphism of Noetherian algebraic spaces. Write $f_{\text{red}} : X_{\text{red}} \to \operatorname{Spec} A_{\text{red}}$ for the morphism of the underlying reduced algebraic spaces.

If f_{red} is an isomorphism, then f is a finite morphism.

Proof. By [CLO12, Corollary 3.1.12], X is a scheme. By [Stacks, Tag 06AD], X is an affine scheme. Write $X = \operatorname{Spec} B$, and let $(x_1, \cdots, x_s)$ be the nilradical of B. Since B is Noetherian, $(x_1, \cdots, x_s)^n = (0)$ for some n. It is harmless to replace A by its image in B. We have

$$B = A + \sum x_i B$$

as an A-module since $A \to B/(x_1, \cdots, x_s)B$ is surjective. Therefore

$$\begin{aligned} B &= A + \sum x_i B \\ &= A + \sum x_i (A + \sum x_j B) \\ &= A + \sum x_i A + \sum x_i x_j A + \ldots + \sum x_{i_1} x_{i_2} \ldots x_{i_n} A; \end{aligned}$$

and thus B is a finite A-module. □

2.7.2. Lemma Let $f : X \to Y$ be a morphism of Noetherian algebraic $\mathbb{Z}/p^a$-stacks that is locally of finite type such that $f(A) : X(A) \to Y(A)$ is an equivalence of categories for all Artinian local $W(\bar{\mathbb{F}}_p)$-algebras A. Then f is an isomorphism.

Proof. Since both X and Y are Noetherian algebraic stacks over $\mathbb{Z}/p^a$, the morphism f is automatically locally of finite presentation. Moreover the set of closed points (=$\bar{\mathbb{F}}_p$-points) is dense in X.

Claim f is étale.

Since $\bar{\mathbb{F}}_p$-points are dense in X, by [Stacks, Tag 02GI], we only need to show f is étale at all $\bar{\mathbb{F}}_p$-points of X. By [Stacks, Tag 02GU], it suffices to show f is étale when localized at stalk rings of $\bar{\mathbb{F}}_p$-points of X. By [Stacks, Tag 02HM], it suffices to show f is formally étale when localized at stalk rings of $\bar{\mathbb{F}}_p$-points of X, which is true by our assumptions.

Claim f is surjective.

f is surjective on $\bar{\mathbb{F}}_p$-points. Since surjectivity descends along surjective morphisms, we can pretend f is a morphism of schemes. By Chevalley's theorem, the complement C of the image of f is constructible. Since Y is Noetherian, C contains a dense open U of its closure $\bar{C}$. If U is non-empty, U must contain a closed point, and we get a contradiction.

Claim f is universally injective.

By [Stacks, Tag 01S4], it suffices to show the diagonal of f is surjective, which can be checked on $\bar{\mathbb{F}}_p$-points by the proof of the previous claim.

By [Stacks, Tag 02LC], a universally injective étale morphism is an open immersion; thus f is a surjective open immersion, that is, an isomorphism. □

2.7.3. Theorem Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of (not necessarily algebraic) stacks over $\operatorname{Spec} \mathbb{Z}/p^a$. Assume

(1) $\mathcal{X}$ is a Noetherian algebraic stack;
(2) $\mathcal{X}_{\text{red}}$ is locally of finite type over $\operatorname{Spec} \mathbb{Z}/p^a$;
(3) The diagonal of $\mathcal{Y}$ is representable by algebraic spaces locally of finite type;
(4) $\mathcal{Y}$ is limit-preserving;
(5) f is fully faithful;

(6) If A is an Artinian local $W(\bar{\mathbb{F}}_p)$-algebra, then $f(A) : \mathcal{X}(A) \to \mathcal{Y}(A)$ is essentially surjective.

Then f is an isomorphism.

Proof. By [Stacks, Tag 04TF], item (1), (3) and (5) ensures that f is relatively representable by algebraic spaces. Let A be a finitely presented $\mathbb{Z}/p^a$-algebra and fix a morphism $\operatorname{Spec} A \to \mathcal{Y}$. We first show $(f_A)_{\mathrm{red}}$ is an isomorphism. Since $\mathcal{X}_{\mathrm{red}}$, $\operatorname{Spec} A$ and the diagonal of $\mathcal{Y}$ are all locally of finite type over $\mathbb{Z}/p^a$, $(f_A)_{\mathrm{red}}$ is locally of finite presentation by the proof of [Stacks, Tag 045G] (note that $\mathcal{X}_{\mathrm{red}} \times_{\mathcal{Y}} \operatorname{Spec} A$ is the base change of $\mathcal{X}_{\mathrm{red}} \times \operatorname{Spec} A$ along the diagonal of $\mathcal{Y}$; we also used [Stacks, Tag 01TB] and [Stacks, Tag 01TX]). By Lemma 2.7.2, $(f_A)_{\mathrm{red}}$ is an isomorphism.

By Lemma 2.7.1, f_A is a finite morphism and hence of finite presentation. Applying Lemma 2.7.2 once again, f_A is an isomorphism. Since $\mathcal{Y}$ is assumed to be limit-preserving, f is an isomorphism ([EG21, Theorem 2.5.1]). □

2.7.4. Corollary Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of (not necessarily algebraic) stacks over $\operatorname{Spec} \mathbb{Z}/p^a$. Assume $\mathcal{Y}$ is isomorphic to a directed colimit of substacks $\varinjlim_m \mathcal{Y}_m$ such that each $\mathcal{Y}_m \to \mathcal{Y}$ is relatively representable by closed immersions. Write $\mathcal{X}_m$ for $\mathcal{Y}_m \underset{\mathcal{Y}}{\times} \mathcal{X}$. Assume

(1) $\mathcal{X}_m$ is a Noetherian algebraic stack;
(2) $\mathcal{X}_{m,\mathrm{red}}$ is locally of finite type over $\operatorname{Spec} \mathbb{Z}/p^a$;
(3) The diagonal of $\mathcal{Y}$ is representable by algebraic spaces locally of finite type;
(4) $\mathcal{Y}$ is limit-preserving;
(5) f is fully faithful;
(6) If A is an Artinian local $W(\bar{\mathbb{F}}_p)$-algebra, then $f(A) : \mathcal{X}(A) \to \mathcal{Y}(A)$ is essentially surjective.

Then f is an isomorphism.

Proof. By Theorem 2.7.3, each $\mathcal{X}_m \to \mathcal{Y}_m$ is an isomorphism. □

2.7.5. Example Corollary 2.7.4 can fail without certain finiteness assumptions. Let $\{a_1, a_2, \cdots\}$ be a non-repeating sequence that exhausts all elements of $\bar{\mathbb{F}}_p$. Let $\mathcal{Y}$ be $\operatorname{Spec} \bar{\mathbb{F}}_p[t,\varepsilon]/(\varepsilon^2)$, and let $\mathcal{X}$ be

$$\varinjlim_m \operatorname{Spec} \bar{\mathbb{F}}_p[t,\varepsilon]/(\varepsilon^2, (t-a_1)^m (t-a_2)^m \cdots (t-a_m)^m \varepsilon).$$

The obvious embedding $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ is not an isomorphism since $\mathcal{X}$ is not a scheme. Note that $\mathcal{X}_{\mathrm{red}} = \mathcal{Y}_{\mathrm{red}} = \operatorname{Spec} \bar{\mathbb{F}}_p[t]$; f is fully faithful since it is an inductive limit of closed immersions; and $f(A)$ is clearly essentially surjective for Artinian local $\bar{\mathbb{F}}_p$-algebras A.

2.7.6. Lemma Let $\widehat{P}$ be a parabolic subgroup of a reductive group $\widehat{G}$, and let T be a $\widehat{P}$-torsor over the field $\bar{\mathbb{F}}_p((u))$. Then T is a trivial $\widehat{P}$-torsor.

Proof. We remark that since the field $\bar{\mathbb{F}}_p((u))$ is not perfect, Steinberg's theorem ([Se02, III.2.3]) only applies to connected reductive groups. See Exercise 3) of [Se02, III.2.1] for an example for a connected unipotent group having non-trivial torsors over $\bar{\mathbb{F}}_p((u))$.

By Exercise 1) of [Se02, III.2.1], the map

$$H^1(\bar{\mathbb{F}}_p((u)), \widehat{P}) \to H^1(\bar{\mathbb{F}}_p((u)), \widehat{G})$$

is injective. For a lack of reference, we include a proof of the exercise. The $\widehat{G}$-torsor $T \times^{\widehat{P}} \widehat{G} \cong \widehat{G}$ is a trivial $\widehat{G}$-torsor. We can thus identify T with a closed subscheme of $\widehat{G}$. By descent, $T/\widehat{P}$ is a form of $\widehat{P}/\widehat{P} = \{*\}$, and is thus isomorphic to $\operatorname{Spec} \bar{\mathbb{F}}_p((u))$. Since $\widehat{G}(\bar{\mathbb{F}}_p((u)))$ acts transitively on the flag variety $(\widehat{G}/\widehat{P})(\bar{\mathbb{F}}_p((u)))$, there exists $g \in \widehat{G}(\bar{\mathbb{F}}_p((u)))$ such that $g(T/\widehat{P}) = \widehat{P}/\widehat{P} \in (\widehat{G}/\widehat{P})(\bar{\mathbb{F}}_p((u)))$. Thus $g \in T(\bar{\mathbb{F}}_p((u)))$ and T admits a rational point. $\square$

2.7.7. Lemma Let A be a finitely presented $\mathbb{Z}_p$-algebra, and let H be a connected smooth affine group over $\operatorname{Spec} \mathbb{Z}_p$.

(1) An H-torsor T over $A((u))$ is a trivial H-torsor if and only if T_{red} is a trivial H-torsor over $A_{\mathrm{red}}((u))$.

(2) Assume H is either reductive, or is a parabolic subgroup of a reductive group. If A is an Artinian $W(\bar{\mathbb{F}}_p)$-algebra, an H-torsor over $A((u))$ is necessarily a trivial H-torsor.

Proof. Part (1) follows from [Ce22, Proposition 6.1.1]. Part (2) follows Lemma 2.7.6 (by part (1), we can replace A by $A/\operatorname{nil}(A)$ and assume A is a finite product of $\bar{\mathbb{F}}_p$.). $\square$

3. Step 1: the moduli of abstract étale φ-modules for split G

In this section, we fix an integer $a > 0$ and a finite flat $\mathbb{Z}/p^a$-algebra Λ. Throughout this section, we choose an endomorphism $\varphi : \Lambda((u)) \to \Lambda((u))$ which is locally contracting of contracting factor $\lambda > 1$ (Definition 2.4.1). For any $\mathbb{Z}_p$-algebra A, write Λ_A for $\Lambda \otimes A$.

Compared to [EG21] and [PR09], we drop the assumption that $\varphi(\Lambda[[u]]) \subset \Lambda[[u]]$, which is the reason for many technical complications.

3.1. Étale φ-modules

3.1.1. Definition Let A be an $\mathbb{Z}/p^a$-algebra. An *étale φ-module with A-coefficients* is a pair (M, ϕ_M) consisting of a finitely generated $\Lambda_A((u))$-module M and an isomorphism of $\Lambda_A((u))$-modules $\phi_M : \varphi^* M \to M$.

Write $\mathrm{Mod}_\varphi^{\text{prét}}(A)$ for the category of projective étale φ-modules with A-coefficients.

We refer the reader to Appendix B for generalities about Tannakian theory. By Lemma B.0.2, the category of projective étale φ-modules is an exact, rigid, symmetric monoidal category.

3.1.2. Definition An *étale φ-module with $\widehat{G}$-structure and A-coefficients* is a faithful, exact, symmetric monoidal functor from ${}^f\mathrm{Rep}_{\widehat{G}}$ to the category $\mathrm{Mod}_\varphi^{\text{prét}}(A)$.

Let $F \in [{}^f\mathrm{Rep}_{\widehat{G}}, \mathrm{Mod}_\varphi^{\text{prét}}(A)]^\otimes$ be an étale ϕ-module with $\widehat{G}$-structure and A-coefficients. Composing F with the forgetful functor $\mathrm{Mod}_\varphi^{\text{prét}}(A) \to \mathrm{Vect}_{\Lambda_A((u))}$, we get an object of $[{}^f\mathrm{Rep}_{\widehat{G}}, \mathrm{Vect}_{\Lambda_A((u))}]^\otimes$, which corresponds to an $\widehat{G}$-torsor over $\Lambda_A((u))$ by Lemma 1.7. We denote by $\mathcal{F}$ the $\widehat{G}$-torsor over $\Lambda_A((u))$ associated to F and we call $\mathcal{F}$ *the underlying $\widehat{G}$-torsor* of F.

3.1.3. Definition A *Kisin lattice with A-coefficients* is a tuple $(\mathcal{K}, j_F, F)$ where F is an étale ϕ-module with $\widehat{G}$-structure and A-coefficients, $\mathcal{K}$ is a $\widehat{G}$-torsor over $\Lambda_A[[u]]$, and $j_F : \mathcal{K}|_{\Lambda_A((u))} \cong \mathcal{F}$ is an isomorphism of $\widehat{G}$-torsors over $\Lambda_A((u))$.

When $\widehat{G} = \mathrm{GL}_d$, we also need the following weaker notion of Kisin lattices.

3.1.4. Definition Let $M \in \mathrm{Mod}_\varphi^{\text{prét}}(A)$. A *weak Kisin lattice* of M is a $\Lambda_A[[u]]$-module $\mathcal{K}$ whose underlying abelian group is a subgroup of M such that $\mathcal{K}[1/u] = M$. Note that $\mathcal{K} \to \mathcal{K}[1/u]$ is an injection if and only if $\mathcal{K}$ is u-torsion-free.

3.1.5. Definition We define the following fppf prestacks over $\mathbb{Z}/p^a$.

Prestack over $\mathbb{Z}/p^a$	Objects over $\operatorname{Spec} A$
${}^w\mathcal{Z}_{\widehat{G}}$	the groupoid of étale φ-module with $\widehat{G}$-structure and A-coefficients
$\mathcal{C}_{\widehat{G}}$	the groupoid of Kisin lattices with A-coefficients

We also define a full substack $\mathcal{Z}_{\widehat{G}}$ of ${}^w\mathcal{Z}_{\widehat{G}}$ by setting

$$\mathcal{Z}_{\widehat{G}} := \text{pro-}({}^w\mathcal{Z}_{\widehat{G}}|_{\mathrm{Aff}_{\mathrm{pf}\,/(\mathbb{Z}/p^a)}}).$$

See [EG21, Subsection 2.5] for unfamiliar notations. The stack $\mathcal{Z}_{\widehat{G}}$ can be characterized as the limit-preserving substack of ${}^w\mathcal{Z}_{\widehat{G}}$ such that $\mathcal{Z}_{\widehat{G}}(A) = {}^w\mathcal{Z}_{\widehat{G}}(A)$ for all finitely presented $\mathbb{Z}/p^a$-algebras.

Let A be a finite type $\mathbb{Z}/p^a$-algebra. An object $F \in \mathcal{Z}_{\widehat{G}}(A)$ is said to be *admissible* if there exists a finite type $\mathbb{Z}/p^a$-algebra B and a scheme-theoretically surjective ([EG21, Definition 3.1.1]) morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that the base changed object F_B has trivial underlying $\widehat{G}$-torsor.

Write $\mathcal{R}_{\widehat{G}}$ for the full substack of $\mathcal{Z}_{\widehat{G}}$ consisting of admissible objects.

3.1.6. Lemma The fppf prestacks $\mathcal{R}_{\widehat{G}}$, $\mathcal{Z}_{\widehat{G}}$, ${}^w\mathcal{Z}_{\widehat{G}}$ and $\mathcal{C}_{\widehat{G}}$ are all stacky.

Proof. It follows from Drinfeld's descent theory. See Lemma A.1.3. □

3.2. Loop groups and a Frobenius stabilization technique

The loop group and positive loop group of $\widehat{G}$ over $\mathbb{Z}/p^a$ are defined as follows

$$\begin{aligned} \mathrm{L}\,\widehat{G}: &\quad A \mapsto \widehat{G}(\Lambda_A((u))) \\ \mathrm{L}^+\,\widehat{G}: &\quad A \mapsto \widehat{G}(\Lambda_A[[u]]), \end{aligned}$$

where A is an $\mathbb{Z}/p^a$-algebra.

The loop group admits a filtration

$$\mathrm{L}^{\le m}\,\widehat{G} : A \mapsto \{X \in \mathrm{L}\,\widehat{G}(A) | i(X), i(X)^{-1} \in u^{-m}\,\mathrm{Mat}_{N\times N}(\Lambda_A[[u]])\}$$

where m is a non-negative integer.

There is a φ-twisted conjugation action of $\mathrm{L}^+\,\widehat{G}$ on $\mathrm{L}\,\widehat{G}$:

$$\begin{array}{ccc} \mathrm{L}\,\widehat{G} \times \mathrm{L}^+\,\widehat{G} & \xrightarrow{m_\varphi} & \mathrm{L}\,\widehat{G} \\ (h, g) & \mapsto & g^{-1}h\varphi(g). \end{array}$$

When $\Lambda[[u]]$ is not φ-stable, the filtration $\mathrm{L}^{\le m}\,\widehat{G}$ is not φ-stable. The goal of this section is to stabilize the filtration.

3.2.1. Lemma There exists an integer $c_\varphi > 0$ such that $\varphi(\Lambda[[u]]) \subset \frac{1}{u^{c_\varphi}}\Lambda[[u]]$.

Proof. Note that if $n \gg 0$, $\varphi(u^n) \in \Lambda[[u]]$. □

3.2.2. Definition Define the stabilized filtration as follows:

$$\mathrm{L}_\varphi^{\le m}\widehat{G} : A \mapsto \{X \in \mathrm{L}\,\widehat{G}(A) | p^{a-i}i(X), p^{a-i}i(X)^{-1} \in u^{-m-c_\varphi i}\,\mathrm{Mat}_{N\times N}(\Lambda[[u]]),\ 0 \le i \le a\}.$$

3.2.3. Lemma (1) $\mathrm{L}_{\varphi}^{\leq m}\widehat{G}$ is a closed subscheme of $\mathrm{L}\,\widehat{G}$.

(2) $\mathrm{L}^{\leq m}\,\widehat{G} \subset \mathrm{L}_{\varphi}^{\leq m}\widehat{G} \subset \mathrm{L}^{\leq m+ac_{\varphi}}\,\widehat{G}$.

(3) $\mathrm{L}\,\widehat{G} = \varinjlim_{m} \mathrm{L}_{\varphi}^{\leq m}\widehat{G}$.

(4) $\mathrm{L}_{\varphi}^{\leq m}\widehat{G}$ is closed under the φ-twisted conjugation action of $\mathrm{L}^{+}\,\widehat{G}$.

Proof. (1): Let $i_s : \mathrm{L}\,\widehat{G} \to L\,\mathrm{Mat}_{n\times n} \xrightarrow{x\mapsto p^s x} L\,\mathrm{Mat}_{n\times n}$ be the multiplication by p^s morphism. Note that $\mathrm{L}_{\varphi}^{\leq m}\widehat{G}$ is the intersection of the pullbacks of closed subschemes along i_s.

(2): Immediate from the definition of $\mathrm{L}_{\varphi}^{\leq m}\widehat{G}$.

(3): It follows from part (1) and part (2).

(4): Note that $\varphi(\mathrm{L}^{+}\,\widehat{G}(A)) \subset \mathrm{L}^{+}\,\widehat{G}(A) + \frac{p}{u^{c_{\varphi}}}\,\mathrm{Mat}_{N\times N}(\Lambda_A[[u]])$ by Lemma 3.2.1. Let $g \in \mathrm{L}^{+}\,\widehat{G}(A)$ and $x \in \mathrm{L}_{\varphi}^{\leq m}\widehat{G}(A)$. Write $\varphi(g) = g_1 + \frac{p}{u^{c_{\varphi}}} g_2$ where $g_1, g_2 \in \frac{1}{u^{c_{\varphi}}}\,\mathrm{Mat}_{N\times N}(A)$. Note that

$$g^{-1}p^{a-i}x\gamma(g) = g^{-1}p^{a-i}xg_1 + g^{-1}\frac{p^{a-i+1}x}{u^{c_{\varphi}}}g_2 \in u^{-m-c_{\varphi}i}\,\mathrm{Mat}_{N\times N}(\Lambda[[u]]), \text{ and}$$

$$g^{-1}p^{a-i}x^{-1}\gamma(g) = g^{-1}p^{a-i}x^{-1}g_1 + g^{-1}\frac{p^{a-i+1}x^{-1}}{u^{c_{\varphi}}}g_2 \in u^{-m-c_{\varphi}i}\,\mathrm{Mat}_{N\times N}(\Lambda[[u]]).$$

So we are done. □

3.3. A generalization of Pappas-Rapoport stacks

3.3.1. Remark In the special case when $\varphi(\Lambda[[u]]) \subset \Lambda[[u]]$, the stacks in this subsection are first considered in [PR09]. In this section, we extend the Pappas-Rapoport construction to the φ-unstable case, which is important when we study cyclotomic (φ, Γ)-modules over ramified extensions E of $\mathbb{Q}_p$. In [EG23], this technical difficulty is bypassed by passing to the induction $\mathrm{Ind}_{\mathrm{Gal}_E}^{\mathrm{Gal}_{\mathbb{Q}_p}}$ while encoding descent data. This technique does not work for general reductive groups G.

The positive loop group has congruence subgroups U_n, defined by

$$U_n : A \mapsto \{X \in \mathrm{L}\,\widehat{G}(A) | i(X) \in 1 + u^n\,\mathrm{Mat}_{N\times N}(A[[u]])\}.$$

3.3.2. Lemma There exists a integer n that only depends on $\varphi : \Lambda((u)) \to \Lambda((u))$ such that $\varphi(U_n) \subset \mathrm{L}^{+}\,\widehat{G}$ for all integers $n > N$.

In particular, the quotient stack $[\mathrm{L}^{\leq m}\,\widehat{G}/_{\varphi}U_n]$ is well-defined for all integers m and all integers $n > N$.

Proof. Choose $N_0 > \frac{1}{d_{\lambda}(\varphi)}$ such that $\varphi(\Lambda[[u]]) \subset \frac{1}{u^{N_0}}\Lambda[[u]]$. Choose $N = \frac{N_0}{\lambda-1}$. □

3.3.3. Lemma Let $n > \max(\frac{2}{\lambda-1}m, N)$ be an integer. The map

$$[\mathrm{L}^{\leq m}\,\widehat{G}/U_n] \to [\mathrm{L}^{\leq m}\,\widehat{G}/_{\varphi}U_n]$$

sending $X \mapsto X$ is an equivalence of categories. (U_n acts by left-translation if notation is not decorated.)

Proof. It is essentially surjective by the definition and it is fully faithful by Lemma 2.4.2. □

The factor group $\mathcal{G}_n := \mathrm{L}^+\widehat{G}/U_n = \mathrm{Res}_{(\mathbb{Z}_p[[u]]/u^n)/\mathbb{Z}_p}(\widehat{G}\times\mathbb{Z}_p[[u]]/u^n)$ is smooth since it is the Weil restriction of a smooth group. Note that $[\mathrm{L}^{\le m}\widehat{G}/U_n] \to [\mathrm{L}^{\le m}\widehat{G}/\mathrm{L}^+\widehat{G}]$ is a $\mathcal{G}_n$-torsor, and $[\mathrm{L}^{\le m}\widehat{G}/\mathrm{L}^+\widehat{G}]$ is a closed subscheme of the affine Grassmannian.

3.3.4. Corollary If $n > N$, the quotient stack

$$[\mathrm{L}\,\widehat{G}/_{\varphi}U_n] = \varinjlim_m [\mathrm{L}^{\le m}\widehat{G}/_{\varphi}U_n]$$

is an Ind-algebraic stack with closed algebraic substacks $[\mathrm{L}^{\le m}\widehat{G}/_{\varphi}U_n]$ that are finitely presented over $\mathbb{Z}/p^a$.

Proof. Fix an integer m, choose $n_m > \max(\frac{2}{\lambda-1}m, N)$. The quotient stack $[\mathrm{L}^{\le m}\widehat{G}/_{\varphi}U_{n_m}] = [\mathrm{L}^{\le m}\widehat{G}/U_{n_m}]$ is representable by a finitely presented scheme. The morphism $[\mathrm{L}^{\le m}\widehat{G}/_{\varphi}U_{n_m}] \to [\mathrm{L}^{\le m}\widehat{G}/_{\varphi}U_n]$ is smooth since it is étale locally represented by the smooth scheme U_n/U_{n_m}. By descent, each $[\mathrm{L}^{\le m}\widehat{G}/_{\varphi}U_n]$ is finitely presented. □

Next we consider the non-standard filtration on $[\mathrm{L}\,\widehat{G}/_{\varphi}U_n]$.

3.3.5. Lemma If $n > N$, the quotient stack

$$[\mathrm{L}\,\widehat{G}/_{\varphi}U_n] = \varinjlim_m [\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}U_n]$$

is an Ind-algebraic stack with closed algebraic substacks $[\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}U_n]$ that are finitely presented over $\mathbb{Z}/p^a$.

Proof. It follows immediately from Corollary 3.3.4 and Lemma 3.2.3. □

3.3.6. Theorem The stack $[\mathrm{L}\,\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$ is representable by an Ind-algebraic stack. We have an Ind-presentation

$$[\mathrm{L}\,\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}] = \varinjlim_m [\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$$

where each $[\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$ is an algebraic stack of finite type over $\mathbb{Z}/p^a$.

Proof. Choose $n > N$. The morphism $[\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}U_n] \to [\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$ is a $\mathcal{G}_n$-torsor. By Lemma 3.3.5, $[\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$ is an algebraic stack of finite type. □

3.3.7. Proposition The stack $\mathcal{C}_{\widehat{G}}$ is isomorphic to the quotient stack $[\mathrm{L}\,\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$.

Proof. Fpqc locally, an object of $\mathcal{C}_{\widehat{G}}$ is a tuple $(\mathcal{K}, j_F, F)$ where $\mathcal{K}$ is the trivial $\widehat{G}$-torsor over $\Lambda_A[[u]]$. Here A is a $\mathbb{Z}/p^a$-algebra. Via j_F, we can regard $\mathcal{F}$ as the trivial $\widehat{G}$-torsor over $\Lambda_A((u))$. The ϕ-structure ϕ_F on F can be represented by an element of $\widehat{G}(\Lambda_A((u)))$. Two such $(\mathcal{K}, j_F, F)$, $(\mathcal{K}, j'_F, F')$ are isomorphic if and only if there is an automorphism g of the trivial $\widehat{G}$-torsor $\mathcal{K}$ such that $g^*\phi_F = \phi_{F'}$. The automorphism g can be represented by an element of $\mathrm{L}^+\widehat{G}(A)$. It is clear the stackification of the quotient prestack $[\mathrm{L}\,\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$ represents $\mathcal{C}_G$. □

3.3.8. Definition Define $\mathcal{C}_{\widehat{G},m}$ to be $[\mathrm{L}^{\le m}_{\varphi}\widehat{G}/_{\varphi}\mathrm{L}^+\widehat{G}]$.

3.3.9. The following diagram illustrates the construction

$$\begin{tikzcd}
{[\mathrm{L}^{\leq m}\widehat{G}/U_{n_m}]} \arrow[r, "\cong"] \arrow[d, "\mathcal{G}_{n_m}\text{-torsor}"] & {[\mathrm{L}^{\leq m}\widehat{G}/_{\varphi}U_{n_m}]} \arrow[d] & & \\
{[\mathrm{L}^{\leq m}\widehat{G}/\mathrm{L}^{+}\widehat{G}]} & {[\mathrm{L}^{\leq m}\widehat{G}/_{\varphi}U_{n}]} \arrow[d, hook] & & \\
& \varinjlim_{m}[\mathrm{L}^{\leq m}\widehat{G}/_{\varphi}U_{n}] \arrow[r, equal] & \varinjlim_{m}[\mathrm{L}_{\varphi}^{\leq m}\widehat{G}/_{\varphi}U_{n}] & {[\mathrm{L}_{\varphi}^{\leq m}\widehat{G}/_{\varphi}U_{n}]} \arrow[l, hook'] \arrow[d, "\mathcal{G}_{n}\text{-torsor}"] \\
& & & {[\mathrm{L}_{\varphi}^{\leq m}\widehat{G}/_{\varphi}\mathrm{L}^{+}\widehat{G}]}
\end{tikzcd}$$

The algebraicity of $\mathcal{C}_{\widehat{G},m}$ ultimately follows from the algebraicity of $[\mathrm{L}^{\leq m}\widehat{G}/\mathrm{L}^{+}\widehat{G}]$.

3.3.10. Theorem (1) The morphism $\mathcal{C}_{\widehat{G}} \to {}^{w}\mathcal{Z}_{\widehat{G}}$ factors through $\mathcal{R}_{\widehat{G}}$. The morphisms

$$\begin{aligned}
&\mathcal{C}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G}}\\
&\mathcal{C}_{\widehat{G},m} \to \mathcal{Z}_{\widehat{G}}\\
&\mathcal{C}_{\widehat{G},m} \to {}^{w}\mathcal{Z}_{\widehat{G}}
\end{aligned}$$

are representable by algebraic spaces, proper, and of finite presentation over $\mathbb{Z}/p^a$.

(2) For each morphism $\operatorname{Spec} A \to \mathcal{R}_{\widehat{G}} \times_{\mathbb{Z}/p^a} \mathcal{R}_{\widehat{G}}$, the change of group morphism

$$\operatorname{Spec} A \underset{\mathcal{R}_{\widehat{G}} \underset{\mathbb{Z}/p^a}{\times} \mathcal{R}_{\widehat{G}}}{\times} \mathcal{R}_{\widehat{G}} \to \operatorname{Spec} A \underset{\mathcal{R}_{\mathrm{GL}_N} \underset{\mathbb{Z}/p^a}{\times} \mathcal{R}_{\mathrm{GL}_N}}{\times} \mathcal{R}_{\mathrm{GL}_N}$$

is a closed immersion. The diagonal morphisms

$$\begin{aligned}
\Delta :&\mathcal{R}_{\widehat{G}} \to \mathcal{R}_{\widehat{G}} \times_{\mathbb{Z}/p^a} \mathcal{R}_{\widehat{G}}\\
\Delta :&\mathcal{Z}_{\widehat{G}} \to \mathcal{Z}_{\widehat{G}} \times_{\mathbb{Z}/p^a} \mathcal{Z}_{\widehat{G}}\\
\Delta :&{}^{w}\mathcal{Z}_{\widehat{G}} \to {}^{w}\mathcal{Z}_{\widehat{G}} \times_{\mathbb{Z}/p^a} {}^{w}\mathcal{Z}_{\widehat{G}}
\end{aligned}$$

are representable by algebraic spaces, affine, and of finite presentation. (The claim holds even if $\widehat{G}$ is replaced by an arbitrary smooth affine possibly disconnected group.)

(3) $\mathcal{C}_{\widehat{G},m}$ is an algebraic stack of finite presentation over $\mathbb{Z}/p^a$ with affine diagonal.

Proof. (0) The morphism $\mathcal{C}_{\widehat{G}} \to {}^{w}\mathcal{Z}_{\widehat{G}}$ factors through the fullsubcategory $[\mathrm{L}\,\widehat{G}/_{\varphi}\mathrm{L}\,\widehat{G}]$ of ${}^{w}\mathcal{Z}_{\widehat{G}}$, which is limit-preserving. It is clear that $[\mathrm{L}\,\widehat{G}/_{\varphi}\mathrm{L}\,\widehat{G}] \subset \mathcal{R}_{\widehat{G}}$.

The same proof works for $\mathcal{R}_{\widehat{G}}$, $\mathcal{Z}_{\widehat{G}}$ and ${}^{w}\mathcal{Z}_{\widehat{G}}$. The morphism $\mathcal{C}_{\widehat{G}} \to \mathcal{R}_{\widehat{G}}$ is relatively represented by a closed subscheme of the twisted affine Grassmannian (see Appendix A.2) for the connected reductive group $\widehat{G}$ by the moduli interpretation. (To elaborate, a morphism $\operatorname{Spec} A \to \mathcal{R}_{\widehat{G}}$ is an étale φ-module with $\widehat{G}$-structure F; and a morphism $\operatorname{Spec} B \to \operatorname{Spec} A \times_{\mathcal{R}_{\widehat{G}}} \mathcal{C}_{\widehat{G}}$ is a Kisin lattice $\mathcal{K}$ together with an identification $\mathcal{K}[1/u] \cong \mathcal{F}_B$. So $\operatorname{Spec} A \times_{\mathcal{R}_{\widehat{G}}} \mathcal{C}_{\widehat{G}}$ classifies all lattices $\mathcal{K}$ inside $\mathcal{F}$.) The filtered pieces $\mathcal{C}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G}}$ are representable by a finite type closed subscheme of the twisted (version of the co-Sato) affine Grassmannian. By [Dri06, Proposition 3.8] or Lemma A.2.5, after a Nisnevich base change, a finite type closed subscheme of the twisted affine Grassmannian becomes a projective scheme.

(2) The same proof works for $\mathcal{R}_{\widehat{G}}$, $\mathcal{Z}_{\widehat{G}}$ and ${}^{w}\mathcal{Z}_{\widehat{G}}$. By [Stacks, 04SI], it is equivalent to the statement that for all $\mathbb{Z}/p^a$-algebra A, and all $F_1, F_2 \in \mathcal{R}_{\widehat{G}}(B)$ the sheaf $\mathrm{Isom}(x,y)$ is representable by an affine scheme of finite presentation over $\operatorname{Spec} A$. [EG21, Theorem 5.4.9(3)] proves this for GL_N (our setup is more general but the same exact argument works; the key point is that an element of $\mathrm{Isom}(x,y)$ is determined by the first few terms in the u-expansion). The lemma follows from the fact that the map $\mathrm{Isom}(F_1,F_2) \to \mathrm{Isom}(F_1(i),F_2(i))$ is a closed immersion. Indeed, there is a pullback diagram

$$\begin{tikzcd} \mathrm{Isom}(F_1,F_2) \arrow[r] \arrow[d] & \mathrm{Isom}(F_1(i),F_2(i)) \arrow[d] \\ (\mathrm{L}\,\widehat{G})_A \arrow[r] & (\mathrm{L}\,\mathrm{GL})_A \end{tikzcd}$$

and the bottom arrow is a closed immersion (if we fix an element $f_0 \in \mathrm{Isom}(F_1,F_2)(A)$, all elements $f \in \mathrm{Isom}(F_1,F_2)(A)$ can be uniquely written as $g \cdot f_0$ for some $g \in \mathrm{L}\,\widehat{G}(A) = \widehat{G}(\Lambda_A((u)))$ because $f \circ f_0^{-1}$ is a $\widehat{G}$-torsor endomorphism over $\Lambda_A((u))$).

(3) The claim that $\mathcal{C}_{\widehat{G},m}$ is of finite presentation with affine diagonal follows from part (1) (2) and Theorem 3.3.6 by the same proof of [EG21, Theorem 5.4.9]. $\square$

3.4. The universal family over $\mathcal{C}_{\widehat{G},m}$

In this subsection, we describe the universal family over $\mathcal{C}_{\widehat{G},m}$.

When $X = \operatorname{Spec} A$ is an affine scheme, an element of $\mathcal{C}_{\widehat{G},m}(\operatorname{Spec} A)$ determines a $\widehat{G}$-torsor over $\operatorname{Spec} \Lambda_A[[u]]$. However, when $X = \operatorname{Spec} A \cup \operatorname{Spec} B$ is a union of two affine schemes, it is not immediately clear what kind of geometric objects $\mathcal{C}_{\widehat{G},m}(\operatorname{Spec} A \cup \operatorname{Spec} B)$ parameterizes.

Write $\operatorname{Spf} \Lambda_A[[u]]$ for the formal scheme which is the u-adic completion of $\operatorname{Spec} \Lambda_A[[u]]$. Note that $\operatorname{Spf} \Lambda_A[[u]]$ is a ringed space whose underlying topological space is identified with $\operatorname{Spec} \Lambda_A$. Recall that a $\widehat{G}$-torsor can be regarded as a tensor functor from ${}^{f}\mathrm{Rep}_{\widehat{G}}$ to the category of finite projective coherent sheaves. So the u-adic completion functor for coherent sheaves induces a u-adic completion functor for $\widehat{G}$-torsors. We still let $X = \operatorname{Spec} A \cup \operatorname{Spec} B$ be a union of affine schemes. The formal schemes $\operatorname{Spf} \Lambda_A[[u]]$ and $\operatorname{Spf} \Lambda_B[[u]]$ canonically glue into a formal scheme $\mathfrak{X}_\Lambda[[u]]$ whose underlying topological space is identified with $X \times \operatorname{Spec} \Lambda =: X_\Lambda$. As a consequence, for each object of $\mathcal{C}_{\widehat{G},m}(X)$, we can associate to it a $\widehat{G}$-torsor over the formal scheme $\mathfrak{X}_\Lambda[[u]]$.

Let $\operatorname{Spec} R \to \mathcal{Z}_{\widehat{G}}$ be a morphism from a Noetherian ring R to $\mathcal{Z}_{\widehat{G}}$. Write X_R for $\mathcal{C}_{\widehat{G},m} \underset{\mathcal{Z}_{\widehat{G}}}{\times} \operatorname{Spec} R$. By the previous paragraph, the morphism $X_R \to \mathcal{C}_{\widehat{C},m}$ defines a $\widehat{G}$-torsor $\mathfrak{K}_R$ over the formal scheme $\mathfrak{X}_{\Lambda_R}[[u]]$ (the u-adic completion of $X_R \times \operatorname{Spec} \Lambda_R[[u]]$). Since $\pi : \mathfrak{X}_{\Lambda_R}[[u]] \to \operatorname{Spf} \Lambda_R[[u]]$ is a proper morphism, by the Grothendieck algebraization theorem, π admits an algebraization $\pi : X_{\Lambda_R}[[u]] \to \operatorname{Spec} \Lambda_R[[u]]$ (see Definition 2.3.6); and by the Grothendieck existence theorem [Stacks, Tag 088F], $\mathfrak{K}_R$ descends to a $\widehat{G}$-torsor $\mathcal{K}_R$ over $X_{\Lambda_R}[[u]]$. For each affine open $\operatorname{Spec} S \subset X_R$, the composition $\operatorname{Spec} S \to X_R \to \operatorname{Spec} R \to \mathcal{Z}_{\widehat{G}}$ defines an étale φ-module with $\widehat{G}$-structure F_S over $\operatorname{Spec} \Lambda_S((u))$. By the moduli interpretations, $\mathcal{K}_R \underset{X_{\Lambda_R}[[u]]}{\times} \operatorname{Spec} \Lambda_S[[u]]$ is a Kisin lattice of F_S. As $\operatorname{Spec} S$ varies, the various F_S glue into an étale φ-module with $\widehat{G}$-structure over $X_{\Lambda_R}((u)) := X_{\Lambda_R}[[u]] \otimes_{\Lambda_R[u]} \Lambda_R[1/u]$.

We record the discussion above in the following proposition.

3.4.1. Proposition Let $\operatorname{Spec} R \to \mathcal{Z}_{\widehat{G}}$ be a morphism from a Noetherian ring R to $\mathcal{Z}_{\widehat{G}}$. Write X_R for $\mathcal{C}_{\widehat{G},m} \underset{\mathcal{Z}_{\widehat{G}}}{\times} \operatorname{Spec} R$. There exists an étale φ-module with $\widehat{G}$-structure F_{X_R} over $X_{\Lambda_R}((u))$, and a $\widehat{G}$-torsor $\mathcal{K}_R$ over $X_{\Lambda_R}[[u]]$ together with an identification $\mathcal{K}_R[1/u] \cong F_{X_R}$.

Moreover, write F_R for the universal family of étale φ-modules with $\widehat{G}$-structure over $\operatorname{Spec} R$, F_{X_R} is the pullback of F_R along the projection $X_{\Lambda_R}((u)) \to \operatorname{Spec} \Lambda_R((u))$.

3.4.2. Corollary Let $\operatorname{Spf} R \to \mathcal{Z}_{\widehat{G}}$ be a morphism from a Noetherian formal scheme to $\mathcal{Z}_{\widehat{G}}$. Write I for the ideal of definition of $\operatorname{Spf} R$.

The morphism $\mathcal{C}_{\widehat{G},m} \underset{\mathcal{Z}_{\widehat{G}}}{\times} \operatorname{Spf} R \to \operatorname{Spf} R$ admits an algebraization $X_R \to \operatorname{Spec} R$. Write $\mathfrak{X}_{\Lambda_R}((u))$ for the I-adic completion of $X_{\Lambda_R}((u))$, and write $\widehat{\jmath} : \mathfrak{X}_{\Lambda_R}((u)) \to X_{\Lambda_R}[[u]]$ for the completed localization map.

There exists an étale φ-module with $\widehat{G}$-structure $\widehat{F}_{X_R}$ over $\mathfrak{X}_{\Lambda_R}((u))$ and a $\widehat{G}$-torsor $\mathcal{K}_R$ over $X_{\Lambda_R}[[u]]$ together with an identification $\widehat{\jmath}^*\mathcal{K}_R \cong F_{X_R}$.

Moreover, write $\widehat{F}_R$ for the universal family of étale φ-modules with $\widehat{G}$-structure over $\operatorname{Spf} R$, $\widehat{F}_{X_R}$ is the pullback of F_R along the projection $\pi : \mathfrak{X}_{\Lambda_R}((u)) \to \operatorname{Spf} \widehat{\Lambda_R((u))}$.

Proof. An algebraization $X_R \to \operatorname{Spec} R$ exists by the Grothendieck algebraization theorem. By taking the inverse limit over R/I^n, the universal Kisin lattice over $X_{\Lambda_{R/I^n}}[[u]]$ glues into a Kisin lattice $\widehat{\mathcal{K}}_R$ over the formal scheme $\varinjlim_n X_{\Lambda_{R/I^n}}[[u]]$. By the Grothendieck existence theorem, the $\widehat{\mathcal{K}}_R$ admits a decompletion $\mathcal{K}_R$ over $X_{\Lambda_R}[[u]]$. The rest of the claim follows from Proposition 3.4.1. $\square$

3.4.3. Proposition Let $\operatorname{Spf} R \to \mathcal{Z}_{\widehat{G}}$ be a morphism from a Noetherian formal scheme to $\mathcal{Z}_{\widehat{G}}$. Write I for the ideal of definition of $\operatorname{Spf} R$. Write $\widehat{F}_R$ for the universal family of étale φ-modules with $\widehat{G}$-structure over $\operatorname{Spf} \widehat{\Lambda_R((u))}$.

If $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction that admits a height theory (Definition 2.6.1) and $X_R \to \operatorname{Spec} R$ is scheme-theoretically dominant (see the previous Corollary for the notation), then there exists an étale φ-module F_R with $\widehat{G}$-structure over $\operatorname{Spec} R$ such that $F_R \underset{\operatorname{Spec} \Lambda_R((u))}{\times} \operatorname{Spec} \widehat{\Lambda_R((u))} = \widehat{F}_R$.

Proof. Regard $\widehat{F}_R$ as a tensor functor in $[{}^f\mathrm{Rep}_{\widehat{G}}, \mathrm{Mod}_\varphi^{\text{prét}}(\operatorname{Spf} R)]$. We first forget the φ-structure, and write $\widehat{\mathcal{F}}_R \in [{}^f\mathrm{Rep}_{\widehat{G}}, \mathrm{Vect}_{\operatorname{Spf} \widehat{\Lambda_R((u))}}]$ for the underlying $\widehat{G}$-torsor of $\widehat{F}_R$. By Corollary 3.4.2, there exists a Kisin lattice $\mathcal{K}_R \in [{}^f\mathrm{Rep}_{\widehat{G}}, \mathrm{Vect}_{\operatorname{Spec} X_{\Lambda_R}[[u]]}]$ over $X_{\Lambda_R}[[u]]$ such that $\pi^*\widehat{\mathcal{F}}_R = \widehat{\jmath}^*\mathcal{K}_R$ as $\widehat{G}$-torsors.

See Definition 2.3.1 for the definition of $\mathrm{Vect}_{\operatorname{Spec} \widehat{\Lambda_R((u))}, \operatorname{Spec} \Lambda_R[[u]], X_{\Lambda_R}[[u]]}$. Define a lax monoidal functor

$$\widehat{\mathcal{F}}\mathcal{K}_R : {}^f\,\mathrm{Rep}_{\widehat{G}} \to \mathrm{Vect}_{\operatorname{Spec} \widehat{\Lambda_R((u))}, \operatorname{Spec} \Lambda_R[[u]], X_{\Lambda_R}[[u]]}$$

by sending $x \mapsto (\mathcal{F}_R(x), \mathcal{K}_R(x))$. By Proposition 2.3.7, there exists a lax monoidal functor

$$\xi : \mathrm{Vect}_{\operatorname{Spec} \widehat{\Lambda_R((u))}, \operatorname{Spec} \Lambda_R[[u]], X_{\Lambda_R}[[u]]} \to \mathrm{Coh}_{\operatorname{Spec} \Lambda_R((u))}.$$

Write $\mathcal{FK}_R := \xi(\widehat{\mathcal{F}}\mathcal{K}_R)$. For each x, $\mathcal{FK}_R(x)$ is equipped with an étale φ-structure, as follows. We first assume the φ-structure $\phi_{\mathcal{K}_R(x)} : \mathcal{K}_R(x)[1/u] \to \mathcal{K}_R(x)[1/u]$ is effective in the sense that $\phi_{\mathcal{K}_R(x)}(\mathcal{K}_R) \subset \mathcal{K}_R$. By the proof of Proposition 2.3.7 (any unfamiliar notation is defined there),

$$\mathcal{FK}_R(x) = j^*\varprojlim_n (j,\pi)_* i_n^*(\widehat{\mathcal{F}}_R(x), \mathcal{K}_R(x)) = j^*\varprojlim_n \kappa_n.$$

Since $\mathcal{K}_R$ admits an effective φ-structure, so does $(j,\pi)_* i_n^*(\widehat{\mathcal{F}}_R(x), \mathcal{K}_R(x))$ for each n. By passing to the the inverse limit and then the localization, $\mathcal{FK}_R(x)$ admits a φ-structure.

Write $T_N = \Lambda[[u]] \subset \Lambda((u))$ for the standard Kisin lattice of the 1-dimensional étale φ-module whose φ-structure is defined by $1 \mapsto u^N$, where $N \gg 0$ is a sufficiently large integer such that the φ-structure on $\mathcal{K}_R \otimes T_N$ is effective. Since $\xi(\mathcal{F}_R(x) \otimes T_N, \mathcal{K}_R(x) \otimes T_N) \otimes T_{-N} = \mathcal{FK}_R(x)$ (by the projection formula), $\mathcal{FK}_R(x)$ admits a canonical φ-structure by transporting the φ-structure on $\xi(\mathcal{F}_R(x) \otimes T_N, \mathcal{K}_R(x) \otimes T_N)$.

Since the inverse limit functor is left-exact, $\varphi^*\mathcal{FK}_R(x) \to \mathcal{FK}_R(x)$ is injective. We haven't used the assumption that φ admits a height theory so far. We claim that $\phi_{\mathcal{FK}_R(x)} : \varphi^*\mathcal{FK}_R(x) \to \mathcal{FK}_R(x)$ is surjective. By the discussion in Paragraph 2.3.8, we can replace $\Lambda[[u]]$ by $\Lambda[[v]]$ when proving the surjectivity of $\phi_{\mathcal{FK}_R(x)}$. Note that $(H4)$ of Definition 2.6.1 allows us to descend the étaleness. By Lemma 3.4.4, there exists an integer $h > 0$ such that $v^h\mathcal{K}_R(x) \subset \varphi^*\mathcal{K}_R(x)$. Since push-forward is left-exact, we have $v^h\pi_*\mathcal{K}_R(x) \subset \pi_*\varphi^*\mathcal{K}_R(x)$. By the flat base change theorem and [EG21, Lemma 5.2.5], $\pi_*\varphi^*\mathcal{K}_R(x) = \varphi^*\pi_*\mathcal{K}_R(x)$. As a consequence $v^h\kappa_n \subset \varphi^*\kappa_n$ for all n. Since inverse limit is left exact, $v^h\varprojlim_n \kappa_n \subset \varprojlim_n \kappa_n$; inverting u, we get $\mathcal{FK}_R(x) \subset \varphi^*\mathcal{FK}_R(x)$, that is, $\phi_{\mathcal{FK}_R(x)}$ is an isomorphism.

So far we've shown for each x, the $\mathcal{FK}_R(x)$ is equipped with an étale φ-module structure, which is clearly functorial in x. So the functor $\mathcal{FK}_R$ is upgraded into a lax monoidal functor from ${}^f\operatorname{Rep}_{\widehat{G}}$ to the category of étale φ-modules over $\Lambda_R((u))$. By Proposition 2.3.7, the I-adic completion of $\mathcal{FK}_R$ is $\widehat{F}_R$. So by Lemma 3.4.5 (the special case $0 \to M = M \to 0 \to 0$), each $\mathcal{FK}_R(x)$ is a finite projective module over $\Lambda_R((u))$. By Lemma 3.4.6, $\mathcal{FK}_R$ is a faithful, exact, strict monoidal functor, and we are done. ∎

3.4.4. Lemma If $\varphi(\Lambda[[v]]) \subset \Lambda[[v]]$ and $\mathcal{K}$ is a Kisin lattice over $X[[v]]$ where X is a quasi-compact Λ-scheme, then there exists an integer $h > 0$ such that $v^h\mathcal{K} \subset \varphi^*\mathcal{K}$, where $\varphi^*\mathcal{K}$ is the injective image of $\Lambda[[v]] \otimes_{\varphi,\Lambda[[v]]} \mathcal{K} \to \varphi^*\mathcal{K}[1/v] \xrightarrow{\phi_{\mathcal{K}[1/v]},\cong} \mathcal{K}[1/v]$.

Proof. Since X is quasi-compact, it suffices to prove the affine case $X = \operatorname{Spec} R$. Let $\{x_1, \dots, x_n\}$ be a set of generators of $\mathcal{K}$. For each x_i, there exists an element v^h such that $u^h\phi(x_i) \in \mathcal{K}$, so we are done. ∎

3.4.5. Lemma Assume $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction. Let R be a complete Noetherian local $\mathbb{Z}_p$-algebra killed by p^a for some integer $a \geq 1$, with maximal ideal $\mathfrak{m}$. Let $M_1 \to M_2 \to M_3$ be a sequence of finitely generated étale φ-modules with R-coefficients.

If the $\mathfrak{m}$-adic completion of $0 \to M_1 \to M_2 \to M_3 \to 0$ is short exact, then $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of projective $\Lambda_R((u))$-modules.

Proof. By Definition 2.5.2, there exists an integer $f > 0$ such that $\varphi^f = \Lambda((u)) \to \Lambda((u))$ is the Λ-linear q-power Frobenius map mod p. By replacing φ by φ^f, we may assume $\varphi \equiv \operatorname{Frob}_q \bmod p$. By [EG21, Theorem 5.5.20], M_i is a projective module, $i = 1, 2, 3$ (note that the running assumption of

[EG21] is $\varphi(\Lambda[[u]]) \subset \Lambda[[u]]$; since $\varphi(\Lambda/p[[u]]) \subset \Lambda/p[[u]]$, we know $M_i \otimes \mathbb{F}_p$ is projective over $\Lambda_{R/p}((u))$; by the last two paragraphs of the proof of [EG21, Theorem 5.5.20] the mod p case implies the general case). By [EG21, Lemma 5.2.14], one can find finite projective étale φ-modules with R-coefficients N_1 and N_3 such that both $M_1 \oplus N_1$ and $M_3 \oplus N_3$ are finite free. By Lemma 2.5.3 and Lemma 2.5.4, $0 \to M_1 \oplus N_1 \to M_2 \oplus N_1 \oplus N_3 \to M_3 \oplus N_3 \to 0$ is a short exact sequence. □

3.4.6. Lemma Let $\mathcal{C}$ be an exact monoidal category, and let R be a complete Noetherian local $\mathbb{Z}/p^a$-algebra with defining ideal I. Let F be a lax monoidal functor from $\mathcal{C}$ to $\mathrm{Vect}_{\mathrm{Spec}\,\Lambda_R((u))}$ such that the I-adic completion $\widehat{F} \in [\mathcal{C}, \mathrm{Vect}_{\mathrm{Spec}\,\widehat{\Lambda_R((u))}}]^{\otimes}$ of F a faithful, exact, strict monoidal functor. If F is equipped with an étale φ-structure, then F is a faithful, exact, strict monoidal functor.

Proof. Faithfulness of F follows from the faithfulness of $\widehat{F}$. Exactness of F is Lemma 3.4.5. Lemma 3.4.5 also implies that F is strict monoidal. □

3.5. Deformation theory

3.5.1. Effective versal rings Fix a Noetherian ring k. Fix a maximal ideal $\mathfrak{m}_k$ of k. Fix a finite field extension $k/\mathfrak{m}_k \to l$. Let $C_k = C_{k,l}$ be the category of pairs (A, θ) where A is an Artinian local k-algebra and $\theta : A_{\mathfrak{m}_A} \to l$ is a k-algebra homomorphism compatible with $k/\mathfrak{m}_k \to l$. Define pro-C_k = pro-$C_{k,l}$ to be the pro-category of C_k. When k is an $\mathbb{Z}_p$-algebra, pro-C_k is equivalent to the category of profinite local k-algebras A, together with a k-algebra homomorphism $A/\mathfrak{m}_A \to l$ compatible with $k/\mathfrak{m}_k \to l$. Let $\widehat{C}_k \subset$ pro-C_k be the full subcategory of Noetherian k-algebras.

Let $\mathcal{F}$ be a groupoid over $\mathrm{Spec}\, k$. Let $x : \mathrm{Spec}\, l \to \mathcal{F}$ be a finite type point. Then x defines a k-algebra homomorphism $k/\mathfrak{m}_k \to l$. Define the groupoid $\widehat{\mathcal{F}}_x$ over C_k by declaring objects lying above $A \in C_k$ to be morphisms $x' \to x$ in $\mathcal{F}$ lying above $\mathrm{Spec}\, l \to \mathrm{Spec}\, A$. By taking inverse limits, we can extend $\widehat{\mathcal{F}}_x$ to a groupoid over $\widehat{C}_k$. We also define the groupoid $\mathcal{F}_x$ over $\widehat{C}_k$ by declaring objects lying above $A \in \widehat{C}_k$ to be morphisms $x' \to x$ in $\mathcal{F}$ lying above $\mathrm{Spec}\, l \to \mathrm{Spec}\, A$. Completion defines a morphism $\mathcal{F}_x \to \widehat{\mathcal{F}}_x$. An object of $\widehat{\mathcal{F}}_x$ is said to be *effective* if it lies in the essential image of $\mathcal{F}_x$.

A morphism $\xi : \mathrm{Spf}\, R \to \widehat{\mathcal{F}}_x$ is said to be a *versal ring* of $\mathcal{F}$ at x if it is formally smooth; ξ is said to be an *effective versal ring* if it defines an effective object of $\widehat{\mathcal{F}}_x(\mathrm{Spf}\, R)$.

3.5.2. Definition Write $\mathcal{Z}_m$ for the fpqc sheafification of the presheaf $A \mapsto \{x \in \mathrm{L}\,\widehat{G}(A) | y^{-1}x\varphi(y) \in \mathrm{L}_{\varphi}^{\leq m}\widehat{G}$ for some $y \in \mathrm{L}\,\widehat{G}(A)\}$.

For ease of notation, write $x \cdot y$ for $x^{-1}y\varphi(x)$. We don't know if $\mathcal{Z}_m$ is (Ind-)algebraic or not. Nonetheless, we are able to understand completion of $\mathcal{Z}_m$ at finite type points.

3.5.3. Lemma If $n \gg m$, the quotient stack $[\mathcal{Z}_m/_{\varphi}U_n]$ is a setoid.

Proof. We want to show that for each $x \in \mathcal{Z}_m$ and $g \in U_n$, if $g \cdot x = x$ then $g = 1$ if $n \gg m$. Choose $h \in \mathrm{L}\,\widehat{G}$ such that $h \cdot x \in \mathrm{L}_{\varphi}^{\leq m}\widehat{G}$. We have $g \cdot x = h^{-1} \cdot (hgh^{-1}) \cdot h \cdot x$. If $g \cdot x = x$, then $(hgh^{-1}) \cdot h \cdot x = h \cdot x$. Since U_n is a normal subgroup of $\mathrm{L}\,\widehat{G}$, $hgh^{-1} \in U_n$. By Lemma 2.4.2, $hgh^{-1} = 1$ if $n \gg m$. So $g = 1$ if $n \gg m$. □

3.5.4. Higher dilatations and congruence subgroups See [PY06, Section 7.2] for the definition of higher dilatations. Note that there is an embedding $i_0 : \mathrm{Res}_{\Lambda/(\mathbb{Z}_p/p^a)} \widehat{G} \to \mathrm{L}^+ \widehat{G}$ by sending $X \mapsto X$. In [PY06], $\mathrm{Res}_{\Lambda/(\mathbb{Z}_p/p^a)} \widehat{G}$ is denoted by $\mathcal{Z}$ and $\mathrm{L}^+ \widehat{G}$ is denote by $\mathcal{X}$. Following the notation of [PY06], set $\Gamma_0 := \mathrm{L}^+ \widehat{G}$, and inductively define Γ_{n+1} to be the dilatation of $i_n(\mathcal{Z})$ on Γ_n and there exists a natural closed immersion $i_{n+1} : \mathcal{Z} \to \Gamma_{n+1}$. By [PY06, Proposition 7.3], the principal congruence group U_m is isomorphic to the m-th dilatation Γ_m of $\mathrm{L}^+ \widehat{G}$.

3.5.5. Lemma The principal congruence groups U_m are formally smooth. Indeed, they are strongly pro-smooth (representable by inverse limits of smooth schemes with affine and finitely presented transition morphisms).

Proof. Since the results in references have Noetherian assumption, we start with showing truncated U_m are smooth. Fix an arbitrary integer $N > 0$. For each $m < N$, the quotient U_m/U_N is the m-th dilatation of $\mathrm{L}^+ \widehat{G}/U_N$ by [PY06, Proposition 7.3]. By [BLR90, Proposition 3, Section 3.2], since both G and $\mathrm{L}^+ \widehat{G}/U_N$ are smooth, U_m/U_N is smooth for any $m < N$. Since $U_m = \varprojlim_N U_m/U_N$, U_m is formally smooth. $\square$

3.5.6. Lemma Let $\widehat{G} \to G'$ be a group homomorphism and a closed immersion of schemes. Write U'_n and $\mathcal{Z}'_m$ for the corresponding subsheaves of $\mathrm{L}\,\widehat{G}'$. If $\widehat{G}$ is a smooth group and $n \gg m$, then the morphism $[\mathcal{Z}_m/_{\varphi}U_n] \to [\mathcal{Z}'_m/_{\varphi}U'_n]$ is relatively representable.

Proof. By Galois/étale descent, we can replace all stacks by their base change from $\mathbb{Z}_p$ to $W(\bar{\mathbb{F}}_p)$ where $W(-)$ is the ring of Witt vectors. By Steinberg's theorem ([Se02, III.2.3]), all $\widehat{G}$-torsors over $\bar{\mathbb{F}}_p((u))$ are trivial $\widehat{G}$-torsors; since nilpotent thickening does not change the classification of $\widehat{G}$-torsors, we can use matrices after base change to $W(\bar{\mathbb{F}}_p)$.

We prove the lemma by applying Grothendieck's representability criterion (or Schlessinger's theorem) for deformation problems. To show a deformation functor F is representable, it suffices to show $F(A) \times_{F(C)} F(B) = F(A \times_C B)$ where A, B, C are Artinian local rings over $W(\bar{\mathbb{F}}_p)/p^a$, together with fixed isomorphisms A/m_A, B/m_A, $C/m_C \cong \bar{\mathbb{F}}_p$.

Write F for $[\mathcal{Z}_m/_{\varphi}U_n]$ and write F' for $[\mathcal{Z}'_m/_{\varphi}U'_n]$. Let R be an Artinian $W(\bar{\mathbb{F}}_p)/p^a$-algebra with local morphism $\mathrm{Spec}\, R \to F'$. The morphism $\mathrm{Spec}\, R \to F'$ defines an element $x_R \in Z'_m(R) \subset \mathrm{L}\,\widehat{G}'(R)$ (well-defined up to φ-twisted conjugation by an element of U'_n). Note that if $n \gg m$, φ-twisted conjugation by an element of U_n or U'_n is equivalent to left translation by an element of U_n or U'_n by Lemma 2.4.2. Fix a local homomorphism $R \to A \times_C B$ and a local morphism $\mathrm{Spec}\, A \times_C B \to F$. The morphism $\mathrm{Spec}\, A \to F \to F'$ defines an element $u_A \in U'_n(A)$ such that $u_A x_A \in \mathrm{L}\,\widehat{G}(A)$. Here $x_A := x_R \otimes_R A$. The morphism $\mathrm{Spec}\, B \to F \to F'$ defines an element $u_B \in U'_n(B)$ such that $u_B x_B \in \mathrm{L}\,\widehat{G}(B)$. Here $x_B := x_R \otimes_R B$. Similarly, the morphism $\mathrm{Spec}\, C \to F \to F'$ defines an element $u_C \in U'_n(C)$ such that $u_C x_C \in \mathrm{L}\,\widehat{G}(C)$. Here $x_C := x_R \otimes_R C$. Moreover, there exists elements $u_{AC}, u_{BC} \in U'_n(C)$ such that $u_{AC}(u_A x_A) \otimes_A C = u_C x_C = u_{BC}(u_B x_B) \otimes_B C$. Note that we must have $u_{AC}, u_{BC} \in U_n(C)$. By Lemma 3.5.5, the group U_n is formally smooth. Thus the elements u_{AC}, u_{BC} have lifts v_A, v_B in $U_n(A)$, $U_n(B)$, respectively. Replacing u_A by $v_A u_A$ and u_B by $v_B u_B$, we may assume $u_A \otimes_A C = u_B \otimes_B C$ (here Lemma 3.5.3 is used). Since the sheaf U_n is pro-representable, there exists an element $u \in U_n(A \times_C B)$ such that $u \otimes A = u_A$ and $u \otimes B = u_B$. Write x' for $x_R \otimes (A \times_C B)$.

Since $\mathrm{L}\,\widehat{G}$ is pro-representable, $\mathrm{L}\,\widehat{G}(A\times_C B) = \mathrm{L}\,\widehat{G}(A)\times_{\mathrm{L}\,\widehat{G}(C)}\mathrm{L}\,\widehat{G}(B)$. Thus $u\ x' \in \mathrm{L}\,\widehat{G}(A\times_C B)$. Since $Z_m = Z'_m \cap \mathrm{L}\,\widehat{G}$, $u\ x' \in Z_m(A\times_C B)$. So we are done. □

The following lemma is a generalization of [EG21, Theorem 5.3.15].

3.5.7. Lemma Let $\mathbb{F}$ be a finite field. For each $x\in\mathcal{Z}_m(\mathbb{F})$, the groupoid $[\widehat{\mathcal{Z}_m/_{\varphi}U_n}]_x$ is representable by a Noetherian complete local ring if $n \gg m$.

Proof. By Lemma 3.5.6, it suffices to consider the $\widehat{G} = \mathrm{GL}_N$-case, for which the same proof as in [EG21, Theorem 5.3.15] works. □

The argument used in the proof of Lemma 3.5.6 can also be used to prove the following lemma, which will be used later.

3.5.8. Lemma Let $\operatorname{Spf} S \to \operatorname{Spf} R$ be a closed immersion of complete local $\mathbb{Z}/p^a$-algebras. By abuse of notation, also denote by U_n the formal completion of U_n at its identity. Suppose U_n acts on $\operatorname{Spf} S$ and $\operatorname{Spf} R$ in a compatible way. Assume $(\operatorname{Spf} R)(\bar{\mathbb{F}}_p) \to [\operatorname{Spf} R/U_n](\bar{\mathbb{F}}_p)$ is surjective.

If $[\operatorname{Spf} R/U_n]$ is pro-representable by a complete local ring, then $[\operatorname{Spf} S/U_n]$ is also pro-representable by a complete local ring.

Proof. It suffices to show $[\operatorname{Spf} S/U_n] \to [\operatorname{Spf} R/U_n]$ is relatively representable. Fix a morphism $x : \operatorname{Spec} T \to [\operatorname{Spf} R/U_n]$ where T is an Artinian $W(\bar{\mathbb{F}}_p)/p^a$-algebra. Write F_S for $[\operatorname{Spf} S/U_n]$ and write F_R for $[\operatorname{Spf} R/U_n]$. Similar to the proof of Lemma 3.5.6, we base change to $W(\bar{\mathbb{F}}_p)/p^a$ and aim to show $\operatorname{Spec} T \times_{F_R} F_S$ preserve pullbacks in the category of Artinian $W(\bar{\mathbb{F}}_p)/p^a$-algebras. Let $T\to A$, $T\to B$, $A\to C$, $B\to C$ be compatible local homomorphisms of local Artinian $W(\bar{\mathbb{F}}_p)/p^a$-algebras; also fix a morphism $\operatorname{Spec} A\times_C B \to F_S$ compatible with $\operatorname{Spec} A\times_C B \to \operatorname{Spec} T \to F_R$.

Choose $x_T \in (\operatorname{Spf} R)(T)$ which lifts x. Write x_A, x_B, x_C, x' for the base change of x to A, B, C, and $A\times_C B$, respectively.

The morphism $\operatorname{Spec} A \to F_S \to F_R$ defines an element $u_A \in U_n(A)$ such that $u_Ax_A \in (\operatorname{Spf} S)(A)$. The morphism $\operatorname{Spec} B \to F_S \to F_R$ defines an element $u_B \in U_n(B)$ such that $u_Bx_B \in (\operatorname{Spf} S)(A)$. The morphism $\operatorname{Spec} C \to F_S \to F_R$ defines an element $u_C \in U_n(C)$ such that $u_Cx_C \in (\operatorname{Spf} S)(C)$. There exists elements $u_{AC}, u_{BC} \in U_n(C)$ such that $u_{AC}(u_Ax_A)\otimes_A C = x_C = u_{BC}(u_Bx_B)\otimes_B C$. Let v_A, v_B be lifts of u_{AC}, u_{BC} in $U_n(A)$, $U_n(B)$, respectively. Replace u_A by v_Au_A and replace u_B by v_Bu_B. Since $[\operatorname{Spf} R/U_n]$ is assumed to be a setoid, we have $u_A\otimes_A C = u_B\otimes_B C$. By the representability of U_n, there exists an element $u\in U_n(A\times_C B)$ such that $u\otimes A = u_A$ and $u\otimes B = u_B$. By the representability of $\operatorname{Spf} S$, $u\ x' \in (\operatorname{Spf} S)(A\times_C B)$. Thus $\operatorname{Spec} T\times_{F_R} F_S$ is relatively representable. □

3.6. Scheme-theoretic image and effective versal rings In this section we define substacks $\mathcal{R}_{\widehat{G},m}$ of $\mathcal{R}_{\widehat{G}}$ by taking the scheme-theoretic image of $\mathcal{C}_{\widehat{G},m}$, and study their effective versal rings.

We start with collecting necessary definitions from [EG21].

3.6.1. Scheme-theoretic image If $f : Y \to Z$ is a quasi-compact morphism of schemes. The *scheme-theoretic image* of f is the closed subscheme of Z defined by the kernel of $\mathcal{O}_Z \to f_*\mathcal{O}_Y$. It is evident from the definition that if $g : X \to Y$ is a quasi-compact morphism, then the scheme-theoretic image of $f\circ g$ is a closed subscheme of the scheme-theoretic image of f. We say f is *scheme-theoretically dominant* if $\mathcal{O}_Z \to f_*\mathcal{O}_Y$ is injective. Let Z_f be the scheme-theoretic image of f. Then it is evident that $Y \to Z_f$ is scheme-theoretic dominant.

Let $R = \varprojlim_n R/I^n$ be a complete Noethetian ring. Let $f : \mathfrak{X} \to \operatorname{Spf} R$ be a quasi-compact morphism of formal schemes. Then the scheme-theoretic image of f is defined to be $\operatorname{Spf} \varprojlim_n S_n$, where $\operatorname{Spec} S_n$ is the scheme-theoretic image of $X_{R/I^n} \to \operatorname{Spec} R/I^n$.

We consider the special case when $f : \mathfrak{X} \to \operatorname{Spf} R$ is a proper morphism. Then f has an algebraization $X \to \operatorname{Spec} R$. Let $\operatorname{Spf} S$ be a scheme-theoretic image of f. By [Stacks, Tag 0A42], the morphism $X \to \operatorname{Spec} R$ factors through $\operatorname{Spec} S$. Hence $X = X \times_{\operatorname{Spec} R} \operatorname{Spec} S$.

Lemma The morphism $f_S : X \to \operatorname{Spec} S$ is scheme-theoretically dominant.

Proof. By the theorem on formal functions, $(f_S)_*\mathcal{O}_X = \varprojlim_n (f_{S_n})_*\mathcal{O}_{X_{S_n}}$. Since the inverse limit functor is at least left exact, $S \to (f_S)_*\mathcal{O}_X$ is injective. □

3.6.2. Definition We define $\mathcal{R}_{\widehat{G},m}$ to be the scheme-theoretic image of $\mathcal{C}_{\widehat{G},m} \to \mathcal{Z}_{\widehat{G}}$ (in the sense of [EG21, Theorem 1.1.1]).

3.6.3. Lemma (1) $\mathcal{R}_{\widehat{G}}$, $\mathcal{Z}_{\widehat{G}}$ and ${}^w\mathcal{Z}_{\widehat{G}}$ admit versal rings at all finite type points.

(2) $\mathcal{R}_{\widehat{G},m}$ admits a Noetherian versal ring at all finite type points.

(3) If $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction that admits a height theory (Definition 2.6.1), $\mathcal{R}_{\widehat{G},m}$ admits an effective Noetherian versal ring at all finite type points.

Proof. (1) By Lemma 2.7.7, the stacks $[\operatorname{L}\widehat{G}/_\varphi \operatorname{L}\widehat{G}]$, $\mathcal{R}_{\widehat{G}}$, $\mathcal{Z}_{\widehat{G}}$ and ${}^w\mathcal{Z}_{\widehat{G}}$ have the same finite type points and their completion at finite type points are the same. The map $f : \operatorname{L}\widehat{G} \to [\operatorname{L}\widehat{G}/_\varphi \operatorname{L}\widehat{G}]$ is formally smooth. As a consequence, for each finite type point $x : \operatorname{Spec}\mathbb{F} \to \mathcal{Z}_{\widehat{G}}$, the complete local ring of $\operatorname{L}\widehat{G}$ at $y \in f^{-1}(x)$ is a versal ring of $\mathcal{Z}_{\widehat{G}}$ at x (by replacing $\mathbb{F}$ by a finite extension $\mathbb{F}'$, we may assume $f^{-1}(x)$ is non-empty).

(2) To help the reader relate to [EG23], we will use the same notation used in [EG21]. Let $x : \operatorname{Spec}\mathbb{F} \to \mathcal{R}_{\widehat{G},m} \to \mathcal{Z}_{\widehat{G}}$ be a finite type point. Denote by $R^\Box$ the (non-Noetherian) versal ring at x constructed in part (1). Write $\mathcal{C}_{\operatorname{Spf} R^\Box} := \operatorname{Spf} R^\Box \times_{\mathcal{Z}_{\widehat{G},m}} \mathcal{C}_{\widehat{G},m}$, and write $R^{\Box,\mathcal{C}}$ for the scheme-theoretic image of $\mathcal{C}_{\operatorname{Spf} R^\Box}$ in $\operatorname{Spf} R^\Box$. See Definition 3.5.2 for the definition of $\mathcal{Z}_m$. By Lemma [EG21, Lemma 5.4.15], $\operatorname{Spf} R^{\Box,\mathcal{C}} \to \operatorname{Spf} R^\Box$ factors through the complete local ring $R^{\Box,F}$ of $\mathcal{Z}_m$ at x. Fix $n \gg m$ as in Lemma 3.5.3 and Lemma 3.5.7. By abuse of notation, also denote by U_n the formal completion of U_n at its identity. The formal group U_n acts on $\operatorname{Spf} R^\Box$ and $\mathcal{C}_{\operatorname{Spf} R^\Box}$ in a compatible way. Since scheme-theoretic image is the universal (minimal) object in the category of closed subschemes over which the morphism factors through, $\operatorname{Spf} R^{\Box,\mathcal{C}}$ is stable under the U_n-action. By Lemma 3.5.8, the morphism $[\operatorname{Spf} R^{\Box,\mathcal{C}}/U_n] \to [\operatorname{Spf} R^{\Box,F}/U_n]$ is relatively representable. By Lemma 3.5.7, $[\operatorname{Spf} R^{\Box,F}/U_n]$ and hence $[\operatorname{Spf} R^{\Box,\mathcal{C}}/U_n]$ is pro-representable by a Noetherian complete local ring. Say $\operatorname{Spf} R^{\mathcal{C}} = [\operatorname{Spf} R^{\Box,\mathcal{C}}/U_n]$ and $\operatorname{Spf} R^F = [\operatorname{Spf} R^{\Box,F}/U_n]$. By [EG21, Lemma 3.2.16], $\operatorname{Spf} R^{\Box,\mathcal{C}} \to (\widehat{R_G})_x$ factors through a versal morphism $\operatorname{Spf} R^{\Box,\mathcal{C}} \to (\widehat{R_{G,m}})_x$. It remains to check that $\operatorname{Spf} R^{\mathcal{C}} \to (\widehat{R_{G,m}})_x$ is also versal. By Galois descent, we can base change to $W(\bar{\mathbb{F}}_p)/p^a$ (see the proof

of Lemma 3.5.6), after which $\mathrm{Spf}\, R^{\square,F} \to \mathrm{Spf}\, R^F$ is clearly surjective as a map of fpqc sheaves. Since

$$\begin{tikzcd} \mathrm{Spf}\, R^{\square,\mathcal{C}} \arrow[r] \arrow[d] & \mathrm{Spf}\, R^{\square,F} \arrow[d] \\ \mathrm{Spf}\, R^{\mathcal{C}} \arrow[r] & \mathrm{Spf}\, R^F \end{tikzcd}$$

is a Cartesian diagram, we see that $\mathrm{Spf}\, R^{\mathcal{C}}$ is a versal ring.

(3) Fix a finite type point $x : \mathrm{Spec}\,\mathbb{F} \to \mathcal{R}_{\widehat{G},m}$. In part (2), we constructed a Noetherian versal ring $R^{\mathcal{C}}$ of $\mathcal{R}_{\widehat{G},m}$ at x.

Since $\mathrm{Spf}\, R^{\square,\mathcal{C}}$ is by definition the scheme-theoretic image of $\mathcal{C}_{\mathrm{Spf}\, R}$ in $\mathrm{Spf}\, R$, $\mathcal{C}_{\mathrm{Spf}\, R^{\square,\mathcal{C}}} \to \mathrm{Spf}\, R^{\square,\mathcal{C}}$ is scheme-theoretically dominant. The following diagram is a Cartesian diagram

$$\begin{tikzcd} \mathcal{C}_{\mathrm{Spf}\, R^{\square,\mathcal{C}}} \arrow[r] \arrow[d] & \mathcal{C}_{\mathrm{Spf}\, R^{\mathcal{C}}} \arrow[d] \\ \mathrm{Spf}\, R^{\square,\mathcal{C}} \arrow[r] & \mathrm{Spf}\, R^{\mathcal{C}} \end{tikzcd}$$

By Lemma 3.5.5, U_n is (the completion at identity of) a pro-smooth group over $\mathrm{Spec}\,\mathbb{Z}/p^a$. By [Stacks, Tag 06HL], formally smooth maps between complete local rings representable by formal power series (which extends to the countably many variables case by the proof of [EG21, Theorem 5.3.15]); thus $\mathcal{O}_{U_n} \cong \mathbb{Z}/p^a[[x_1, x_2, \dots]]$ and $R^{\square,\mathcal{C}} = R^{\mathcal{C}}[[x_1, x_2, \dots]] = R^{\mathcal{C}} \,\widehat{\otimes}\, \mathcal{O}_{U_n}$. So we conclude that $\mathcal{C}_{\mathrm{Spf}\, R^{\mathcal{C}}} \to \mathrm{Spf}\, R^{\mathcal{C}}$ is also scheme-theoretically dominant.

By Proposition 3.4.3, the Noetherian-ness of R guarantees that R is an effective versal ring. $\square$

3.7. Representability of $\mathcal{R}_{\widehat{G}}$

3.7.1. Theorem Assume $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction that admits a height theory (Definition 2.6.1). Then $\mathcal{R}_{\widehat{G},m}$ is an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathbb{Z}/p^a$. The morphism $\mathcal{C}_{\widehat{G},m} \to \mathcal{Z}_{\widehat{G}}$ factors as

$$\mathcal{C}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G}} \to \mathcal{Z}_{\widehat{G}},$$

with the first morphism being a proper surjection, and the second a closed immersion.

Proof. It follows from [EG21, Theorem 1.1.1] (whose assumptions are checked in Theorem 3.3.10 and Lemma 3.6.3), except that [EG21, Theorem 1.1.1] only implies $\mathcal{R}_{\widehat{G},m}$ is an algebraic stack locally of finite presentation. By the definition of scheme-theoretic image, $\mathcal{C}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G},m}$ is scheme-theoretically dominant. Since $\mathcal{C}_{\widehat{G},m}$ is of finite presentation, we can choose a finitely presented smooth cover $\mathrm{Spec}\, S \to \mathcal{C}_{\widehat{G},m}$. Note that the composition $\mathrm{Spec}\, S \to \mathcal{R}_{\widehat{G},m}$ is scheme-theoretically dominant and defines an admissible family of family of étale φ-modules with $\widehat{G}$-structure; therefore $\mathcal{R}_{\widehat{G},m} \to \mathcal{Z}_{\widehat{G}}$ factors through $\mathcal{R}_{\widehat{G}}$. It remains to show $\mathcal{R}_{\widehat{G},m}$ is quasi-compact and quasi-separated. Since $\mathcal{C}_{\widehat{G},m}$ is quasi-compact and $\mathcal{C}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G},m}$ is surjective, it follows from [Stacks, 050X] that $\mathcal{R}_{\widehat{G},m}$ is quasi-compact. By Theorem 3.3.10, $\mathcal{R}_{\widehat{G},m}$ has affine diagonal and thus is quasi-separated, as required. $\square$

3.7.2. Corollary Assume $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction that admits a height theory (Definition 2.6.1). The following diagram is commutative

$$\begin{tikzcd} \mathcal{C}_{\widehat{G},m} \arrow[r] \arrow[d] & \mathcal{C}_{\mathrm{GL}_N,m} \arrow[d] \\ \mathcal{R}_{\widehat{G},m} \arrow[r] \arrow[d] & \mathcal{R}_{\mathrm{GL}_N,m} \arrow[d] \\ \mathcal{R}_{\widehat{G}} \arrow[r] & \mathcal{R}_{\mathrm{GL}_N} \end{tikzcd}$$

where the bottom square is Cartesian, and

$$\mathcal{C}_{\widehat{G},m} \to \mathcal{C}_{\mathrm{GL}_N,m} \underset{\mathcal{R}_{\mathrm{GL}_N,m}}{\times} \mathcal{R}_{\widehat{G},m}$$

is a closed immersion. Moreover the morphism $\mathcal{R}_{\widehat{G},m} \to \mathcal{R}_{\mathrm{GL}_N,m}$ is relatively representable by algebraic spaces of finite type over $\mathbb{Z}/p^a$.

Proof. The morphism $\mathcal{C}_{\widehat{G},m} \to \mathcal{R}_{\mathrm{GL}_N}$ factors through $\mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\mathrm{GL}_N}} \mathcal{R}_{\mathrm{GL}_N,m}$, which is a closed substack of $\mathcal{R}_{\widehat{G}}$. By [EG21, Lemma 3.2.31], $\mathcal{R}_{\widehat{G},m}$ is a closed substack of $\mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\mathrm{GL}_N}} \mathcal{R}_{\mathrm{GL}_N,m}$, which forces $\mathcal{R}_{\widehat{G},m} = \mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\mathrm{GL}_N}} \mathcal{R}_{\mathrm{GL}_N,m}$. To show $\mathcal{C}_{\widehat{G},m} \to \mathcal{C}_{\mathrm{GL}_N,m} \underset{\mathcal{R}_{\mathrm{GL}_N,m}}{\times} \mathcal{R}_{\widehat{G},m}$ is a closed immersion, we argue by descent and show that for a chosen smooth cover $\operatorname{Spec} B \to \mathcal{R}_{\widehat{G},m}$, the base-changed morphism

$$\mathcal{C}_{\widehat{G},m} \underset{\mathcal{R}_{\widehat{G},m}}{\times} \operatorname{Spec} B \to \mathcal{C}_{\mathrm{GL}_N,m} \underset{\mathcal{R}_{\mathrm{GL}_N,m}}{\times} \operatorname{Spec} B$$

is a closed immersion; since both schemes above are closed subschemes of the twisted affine Grassmannian (for $\widehat{G}$ and GL_N, respectively) by the proof of part (1) of Theorem 3.3.10, the claim follows from Lemma A.2.4.

Denote by f the projection $\mathcal{R}_{\widehat{G},m} = \mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\mathrm{GL}_N}} \mathcal{R}_{\mathrm{GL}_N,m} \to \mathcal{R}_{\mathrm{GL}_N,m}$. By [Stacks, Tag 04T2], f is representable by algebraic stacks. It is easy to see that f is representable by setoids, and hence by algebraic spaces. $\square$

3.7.3. Theorem Assume $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction that admits a height theory (Definition 2.6.1). For $m \geq 0$, the natural morphism $\mathcal{R}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G},m+1}$ is a closed immersion. We have an isomorphism

$$\varinjlim_{m\geq 1} \mathcal{R}_{\widehat{G},m} = \mathcal{R}_{\widehat{G}}.$$

In particular, $\mathcal{R}_{\widehat{G}}$ is a limit-preserving Ind-algebraic stack over $\mathbb{Z}/p^a$.

Proof. By Theorem 3.7.1, the composite and the second morphism of the chain of morphisms $\mathcal{R}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G},m+1} \to \mathcal{R}_{\widehat{G}}$ are closed immersions, the first morphism is also a closed immersion.

It remains to show the natural morphism $\varinjlim_{m\geq 1} \mathcal{R}_{\widehat{G},m} \to \mathcal{R}_{\widehat{G}}$ is an isomorphism. Since $\mathcal{R}_{\widehat{G}}$ is by definition limit-preserving, it suffices to show that for any finitely presented $\mathbb{Z}/p^a$-algebra, there exists an integer m such that the base changed morphism

$$t_A : \mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\widehat{G}}} \operatorname{Spec} A \to \operatorname{Spec} A$$

is an isomorphism. By the definition of $\mathcal{R}_{\widehat{G}}$, there exists a scheme-theoretically dominant, finitely presented morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $\operatorname{Spec} B \to \mathcal{R}_{\widehat{G}}$ defines an étale φ-module with $\widehat{G}$ structure F whose underlying $\widehat{G}$-torsor is trivial. Therefore $\operatorname{Spec} B \to \mathcal{R}_{\widehat{G}}$ factors through $\mathcal{C}_{\widehat{G}}$ and thus factors through $\mathcal{C}_{\widehat{G},m}$ for some m. By [Stacks, Tag 0CPV], the scheme-theoretic image of $\operatorname{Spec} B$ in $\operatorname{Spec} A$ factors through $\mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\widehat{G}}} \operatorname{Spec} A$. But since $\operatorname{Spec} B \to \operatorname{Spec} A$ is scheme-theoretically dominant, $\mathcal{R}_{\widehat{G},m} \times_{\mathcal{R}_{\widehat{G}}} \operatorname{Spec} A = \operatorname{Spec} A$. □

3.7.4. Proposition Assume $\varphi : \Lambda((u)) \to \Lambda((u))$ is a Frobenius contraction that admits a height theory (Definition 2.6.1).

The inclusion $f : \mathcal{R}_{\widehat{G}} \to \mathcal{Z}_{\widehat{G}}$ is an isomorphism (i.e. an equivalence of 2-categories).

Proof. By Lemma 2.7.7, f is an isomorphism if and only if $f \otimes_{\mathbb{Z}/p^a} \mathbb{Z}/p$ is an isomorphism (note that since both $\mathcal{R}_{\widehat{G}}$ and $\mathcal{Z}_{\widehat{G}}$ are limit-preserving, it suffices to check that over finitely presented test objects f is essentially surjective). Thus it is harmless to assume $a = 1$ in the rest of the proof. By Lemma [EG21, Proposition 5.4.7] (whose proof relies on Grothendieck's generic flatness criterion and is valid when A is finitely presented), $\mathcal{R}_{\mathrm{GL}_N} = \mathcal{Z}_{\mathrm{GL}_N}$. Define

$$\mathcal{Z}_{\widehat{G},m} := \mathcal{Z}_{\widehat{G}} \underset{\mathcal{R}_{\mathrm{GL}_N}}{\times} \mathcal{R}_{\mathrm{GL}_N,m}.$$

Note that $\mathcal{Z}_{\widehat{G},m} \to \mathcal{Z}_{\widehat{G}}$ is relatively representable by closed immersions because it is the base change of a closed immersion. By Corollary 3.7.2, $\mathcal{R}_{\widehat{G},m} = \mathcal{Z}_{\widehat{G},m} \underset{\mathcal{Z}_{\widehat{G}}}{\times} \mathcal{R}_{\widehat{G}}$. By Lemma 2.7.7, the assumptions of Corollary 2.7.4 are satisfied and thus f is an isomorphism. □

3.7.5. Corollary Let A be a finitely presented $\mathbb{Z}/p^a$-algebra. An étale φ-module with $\widehat{G}$-structure and A-coefficients is necessarily admissible.

Proof. It is a reformulation of Proposition 3.7.4. □

4. Cyclotomic (φ, Γ)-modules

In this section, we assume $K/\mathbb{Q}_p$ is a p-adic field, and E/K is a finite, tamely ramified, Galois extension. Write G_K for Gal_K.

We write $K(\zeta_{p^\infty})$ to denote the extension of K obtained by adjoining all p-power roots of unity. Denote by $\widehat{(-)}$ the p-adic completion of $(-)$, and denote by $(-)^\flat$ the tilt of $(-)$.

If E is a perfect ring, denote by $[-] : E \to W(E)$ the (multiplicative) Teichmuller lift map.

Fix a compatible system of p-power roots of unity $\varepsilon \in (\widehat{\mathbb{Q}_p(\zeta_{p^\infty})})^\flat$. Denote by $\mathbb{A}'_{\mathbb{Q}_p}$ the image of

$$\begin{aligned} \widehat{\mathbb{Z}_p((T'))} &\hookrightarrow W((\widehat{\mathbb{Q}_p(\zeta_{p^\infty})})^\flat) \\ T' &\mapsto [\varepsilon] - 1 \end{aligned}$$

Set $\mathbb{B}'_{\mathbb{Q}_p} := \mathbb{A}_{\mathbb{Q}_p}[1/p]$, which is a discretely valued field admitting p as a uniformizer. The residue field of $\mathbb{B}'_{\mathbb{Q}_p}$ is $\mathbb{E}'_{\mathbb{Q}_p} = \mathbb{F}_p((\pi'))$ where $\pi' = \varepsilon - 1$.

Denote by H'_K the kernel of the cyclotomic character $\chi_{\mathrm{cyc}} : G_K \to \mathbb{Z}_p^\times$. Denote by $\mathbb{E}'$ the separable closure of $\mathbb{E}'_{\mathbb{Q}_p}$ in $\mathbb{C}^\flat$. Define $\mathbb{E}'_K := (\mathbb{E}')^{H'_K}$, and let $\mathbb{B}'_K$ be the unique unramified extension of $\mathbb{B}'_{\mathbb{Q}_p}$ in

$W(\mathbb{C}^\flat)[1/p]$ with residue field $\mathbb{E}'_K$. Set $\mathbb{A}'_K := \mathbb{B}'_K \cap W(\mathbb{C}^\flat)$, which is a discrete valuation ring, with $\mathbb{A}'_K/p\mathbb{A}'_K = \mathbb{E}'_K$.

We need the following standard fact.

4.0.1. Lemma Denote by k'_∞ the residue field of $K(\zeta_{p^\infty})$.

(1) If $\pi'_K \in \mathbb{E}'_K$ is a uniformizer of $\mathbb{E}'_K$, then $\mathbb{E}'_K = k'_\infty((\pi'_K))$.

(2) If $T'_K \in \mathbb{A}'_K$ is a lift of a uniformizer π'_K of $\mathbb{E}'_K$, then $\mathbb{A}'_K = \widehat{W(k'_\infty)((T'_k))}$.

Proof. (1) See [Be10, Proposition 15.6].

(2) See [Be10, Proposition 17.3]. □

4.1. Tame extensions of Fontaine's rings

4.1.1. Lemma (Krasner's Lemma) Let F be a complete nonarchimedean field, and E is a closed subfield of F. Let $\alpha, \beta \in F$ with α separable over E. Assume $|\beta - \alpha| < |\alpha' - \alpha|$ for all conjugates of α' of α over E, $\alpha' \neq \alpha$. Then $\alpha \in E(\beta)$.

Proof. It is copied from [FO10, Proposition 3.1]. □

The following lemma should be standard (and is mentioned before Theorem 1.4 in [Be14]). We include a proof here for lack of reference.

4.1.2. Lemma Let π'_K be an arbitrary uniformizer of $\mathbb{E}'_K$, and let $T'_K \in \mathbb{A}'_K$ be a lift of π'_K. After replacing E by an unramified extension of E, $(\pi'_K)^{1/n} \in \mathbb{E}'_E$, $(T'_K)^{1/n} \in \mathbb{A}'_E$ and $(\pi'_K)^{1/n}$ is a uniformizer of $\mathbb{E}'_E$.

Proof. Write λ for a primitive n-th root of unity, where n is the ramification index of $\mathbb{E}'_E/\mathbb{E}'_K$. Write $\mathbb{F}$ for the residue field of $\mathbb{E}'_E$. Say $\mathbb{E}'_E[(\pi'_K)^{1/n}] = \mathbb{F}'((\pi))$ where $\mathbb{F}'$ is a finite field. After replacing E by an unramified extension, we may assume $\mathbb{E}'_E[(\pi'_K)^{1/n}] = \mathbb{F}((\pi))$ and $\lambda \in \mathbb{F}$.

Let π'_E be an arbitrary uniformizer of $\mathbb{E}'_E$. It is harmless to assume π'_E, $\pi'_K \in \mathbb{F}[[\pi]]$. Note that $|\pi'^n_E| = |\pi'_K|$. So $\frac{\pi'_E}{(\pi'_K)^{1/n}} \in \mathbb{F}[[\pi]]^\times$.

After replacing π'_E by $c\pi'_E$ (for some $c \in \mathbb{F}$), $\frac{\pi'_E}{(\pi'_K)^{1/n}} = 1 + \pi f(\pi) \in 1 + \pi\mathbb{F}[[\pi]]$. So $|\pi'_E - (\pi'_K)^{1/n}| = |(\pi'_K)^{1/n}\pi f(\pi)| < |(\pi'_K)^{1/n}|$.

On the other hand, $|\lambda^i(\pi'_K)^{1/n} - (\pi'_K)^{1/n}| = |(\pi'_K)^{1/n}|$ for all integers $0 < i < n$. By Krasner's Lemma, $(\pi'_K)^{1/n} \in \mathbb{E}'_E$. So $(\pi'_K)^{1/n}$ is a uniformizer of $\mathbb{E}'_E$.

Next we consider the characteristic 0 lift. Let T'_E be an arbitrary lift of $(\pi'_K)^{1/n}$ in $\mathbb{A}'_E$. We have $|T'_E - (T'_K)^{1/n}|_p < 1$. On the other hand, $|(T'_K)^{1/n} - \lambda^i(T'_K)^{1/n}|_p = 1$ for all $0 < i < n$ since the reduction mod p is $(\lambda^i - 1)(\pi'_K)^{1/n} \neq 0$. By Krasner's Lemma, $(T'_K)^{1/n} \in \mathbb{A}'_E$. □

4.1.3. Corollary After possibly replacing E by an unramified extension, we have

$$\mathbb{E}'_E = (k_E)'_\infty(((\pi'_K)^{1/n})),$$
$$\mathbb{A}'_E = \widehat{W((k_E)'_\infty)(((T'_K)^{1/n}))}.$$

where $\pi'_K \in \mathbb{E}'_K$ and $T'_K \in \mathbb{A}'_K$.

Proof. Combine Lemma 4.0.1 and Lemma 4.1.2. □

4.2. Frobenius and Galois structures on Fontaine's rings

The rings $W(\mathbb{C}^\flat)[1/p]$ and $\mathbb{C}^\flat$ are naturally endowed with a φ-structure and a $G_{\mathbb{Q}_p}$-action that commute with each other. The inclusions $\mathbb{A}'_K, \mathbb{B}'_K \subset W(\mathbb{C}^\flat)[1/p]$ and $\mathbb{E}'_K \subset \mathbb{C}^\flat$ induces natural φ-structures and G_K-actions on these rings.

Recall that $\mathbb{A}'_{\mathbb{Q}_p} = \widehat{\mathbb{Z}_p((T'))}$, where $T' = [\varepsilon] - 1$. We have

$$\begin{aligned} \varphi(T') &= (1+T')^p - 1 \\ \gamma(T') &= (1+T')^{\chi_{\mathrm{cyc}}(\gamma)} - 1,\ \gamma \in G_{\mathbb{Q}_p}. \end{aligned}$$

where $(1+T')^c$ for general $c \in \mathbb{Z}_p$ is computed by (p-adic limits of) the binomial expansion.

Note that H'_K is a subgroup of $\mathbb{Z}_p^\times$. Write H_K for the pro-p quotient of H'_K, and set $\Delta_K := \ker(H'_K \to H_K)$. Write K_∞/K for the Galois extension with Galois group H_K. Define the following rings:

$$\begin{aligned} \mathbb{A}_K &:= (\mathbb{A}'_K)^{\Delta_K} \\ \mathbb{B}_K &:= (\mathbb{B}'_K)^{\Delta_K} \\ \mathbb{E}_K &:= (\mathbb{E}'_K)^{\Delta_K}. \end{aligned}$$

The G_K-action on $\mathbb{A}_K$, $\mathbb{B}_K$ and $\mathbb{E}_K$ factor through H_K.

The structure of $\mathbb{A}_{\mathbb{Q}_p}$ can be described as follows: if we set

$$T = -p + \sum_{a \in \mathbb{F}_p} [\varepsilon]^{[a]} = \sum_{a \in \mathbb{F}_p^\times} ([\varepsilon]^{[a]} - 1),$$

then $\mathbb{A}_{\mathbb{Q}_p} = \widehat{\mathbb{Z}_p((T))}$. It is clear that $\mathbb{A}^+_{\mathbb{Q}_p} := \mathbb{Z}_p[[T]]$ is both φ-stable and $G_{\mathbb{Q}_p}$-stable. If we set $\pi_{\mathbb{Q}_p} = T \bmod p$, then $\mathbb{E}_{\mathbb{Q}_p} = \mathbb{F}_p((\pi_{\mathbb{Q}_p}))$.

4.2.1. Lemma If E/K is Galois and tamely ramified, then

(1) $\mathbb{E}_E/\mathbb{E}_K$ is a Galois extension with $\mathrm{Gal}(\mathbb{E}_E/\mathbb{E}_K) = \mathrm{Gal}(E/K)$;

(2) $\mathbb{B}_E/\mathbb{B}_K$ is an unramified Galois extension with $\mathrm{Gal}(\mathbb{B}_E/\mathbb{B}_K) = \mathrm{Gal}(E/K)$.

(3) The finite étale map $\mathbb{A}_K \to \mathbb{A}_E$ is a Galois cover in the sense of [Stacks, 03SF].

Proof. (1) It is a special case of [Fon90, A.3.1.8]. Note that $E(K)$ (the notation in [Fon90]) is not exactly the same as $\mathbb{E}_K$ when K is wildly ramified over $\mathbb{Q}_p$. Write p^r for the wild ramification index of K over $\mathbb{Q}_p$. Then $E(K) = \varphi^{-r}(\mathbb{E}_K)$, $E(E) = \varphi^{-r}(\mathbb{E}_E)$. Since $\mathbb{C}^\flat$ is a perfect field, $E(E)/E(K)$ is Galois of extension degree n if and only if $\mathbb{E}_E/\mathbb{E}_K$ is Galois of extension degree n.

(2) It follows immediately from (1). Since the residue field is an infinite field, being unramified does not guarantee being Galois in general.

(3) It follows immediately from (2). $\square$

4.2.2. Lemma Assume E is tamely ramified over $\mathbb{Q}_p$ with ramification index e.

After possibly replacing E by an unramified extension of E, $\varphi : \mathbb{A}_E/p^a \to \mathbb{A}_E/p^a$ admits an height theory (Definition 2.6.1) for all integers $a > 0$.

Proof. The same exact proof as in Lemma 4.1.2 shows that $T^{1/e} \in \mathbb{A}_E$ after replacing E by an unramified extension. Note that the only input we need in the proof of Lemma 4.1.2 is that $|\pi_E^e| = |\pi_{\mathbb{Q}_p}|$

where π_E is an arbitrary uniformizer of $\mathbb{E}_E$, which is guaranteed by Lemma 4.2.1(1). Therefore $\mathbb{A}_E/p^a = W(\mathbb{F})/p^a((T^{1/e}))$ and T is a height theory.

It is a standard fact that the Frobenius map $\varphi : \mathbb{E}_E \to \mathbb{E}_E$ is the p-power map. Note that $\mathbb{A}_E = W(\widehat{(k_E)_\infty)}((T_E))$. After replacing φ by φ^f where p^f is the cardinality of $(k_E)_\infty$, φ acts trivially on $W((k_E)_\infty)$.

It remains to show $\mathbb{A}_{\mathbb{Q}_p} \otimes_{\varphi,\mathbb{A}_{\mathbb{Q}_p}} \mathbb{A}_E = \mathbb{A}_E$, which is immediate from the equivalence of categories between Galois representations and étale (φ,Γ)-modules. Note that $\mathbb{A}_E$ corresponds to the $\mathrm{Gal}_{\mathbb{Q}_p}$-representation which is the induction of the trivial representation of Gal_E. ☐

4.2.3. Étale (φ,Γ)-modules

Rings and modules are endowed with *the natural topology* as explained in [EG23, Appendix D].

In this subsection, we are discussion (φ,Γ)-modules for the field K. Denote by Γ the pro-p group H_K and fix once for all a topological generator $\gamma \in \Gamma$. Note that $\varphi\gamma = \gamma\varphi$ as operators on $\mathbb{A}_K$.

4.2.4. Definition Let A be an $\mathbb{Z}/p^a$-algebra.

- An *étale (φ,γ)-module* (M,φ_M,γ_M) with A-coefficients is an étale φ-module (M,ϕ_M) with A-coefficients equipped with a γ-semilinear bijection $\gamma_M : M \to M$ which commutes with ϕ_M.
- An *étale (φ,Γ)-module with A-coefficients* is an étale (φ,γ)-module (M,ϕ_M,γ_M) with A-coefficients such that the map $(\gamma_M - 1)$ is topologically nilpotent.

4.2.5. Lemma The category of étale (φ,γ)-modules (étale (φ,Γ)-modules, resp.) with A-coefficients is an exact, rigid, symmetric monoidal category.

Proof. The statement for étale (φ,γ)-modules follows from an argument similar to Lemma B.0.2. The unit object $\mathbb{A}_K$ of the Tannakian category of étale (φ,γ)-modules is clearly an étale (φ,Γ)-modules. It remains to show the tensor product of two étale (φ,Γ)-modules is an étale (φ,Γ)-module, which follows immediately from the definitions. ☐

The following theorem is standard.

4.2.6. Theorem Let A be an Artinian $\mathbb{Z}/p^a$-algebra. The category of finite projective étale (φ,Γ)-modules with A-coefficients is equivalent to the category of Galois representations $G_K \to \mathrm{Aut}_A(V)$ where V is a finite projective A-module.

4.2.7. Fontaine's rings decorated with coefficients

Let A be an $\mathbb{Z}_p$-algebra. Let $\mathbb{A}_K = W(\widehat{k_\infty)((T}_K))$ be a presentation of the ring $\mathbb{A}_K$.

4.2.8. Definition The ring $\mathbb{A}_{K,A}$ is defined to be $(W(k_\infty)\widehat{\otimes_{\mathbb{Z}_p}} A)((T_K))$.

4.2.9. Definition Let H be a smooth affine group over $\mathbb{Z}_p$. An H-torsor T over $\mathbb{A}_{K,A}$ is said to be *locally trivial* if there exists a flat cover $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ such that $T \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{K,B}$ is a trivial H-torsor over $\mathbb{A}_{K,B}$.

5. Step 2: the moduli of cyclotomic étale φ-modules for non-split G

In this section, we assume E/K is a tamely ramified Galois extension. Note that both $\mathbb{A}_K/p^a$ and $\mathbb{A}_E/p^a$ are rings with φ-structure that satisfy the running assumption of Section 3.

Denote by $\mathcal{R}_{E,\widehat{G}}$ ($\mathcal{R}_{K,\widehat{G}}$, resp.) the moduli stack defined in 3.1.5 that uses $\mathbb{A}_E/p^a$ ($\mathbb{A}_K/p^a$, resp.) as the φ-ring.

5.1. Non-admissibility of étale φ-modules in the ramified case

See Definition 3.1.5 for the definition of admissibility. In Section 3, we establish the Ind-algebraicity of the moduli stack of étale φ-modules with $\widehat{G}$-structure by showing that it coincides with the moduli stack of admissible étale φ-modules with $\widehat{G}$-structure (Proposition 3.7.4), which, in turn, can be identified with the scheme-theoretic image of its Kisin resolution (Theorem 3.7.3).

In this subsection, we explain that if G is ramified, then étale φ-modules with ${}^LG$-structure that corresponds to L-parameters are never admissible; as a consequence, the scheme-theoretic image of the moduli of Kisin lattices in the moduli of étale φ-modules with ${}^LG$-structure is disjoint from the moduli stack of L-parameters.

Let $\rho : \mathrm{Gal}_K \to {}^LG(\bar{\mathbb{F}}_p)$ be an L-parameter. Then the composition $\mathrm{Gal}_K \xrightarrow{\rho} {}^LG(\bar{\mathbb{F}}_p) \to \mathrm{Gal}(E/K)$ is the quotient homomorphism. Denote by (F, ϕ_F) the cyclotomic étale φ-module with ${}^LG$-structure which comes from ρ by Fontaine's monoidal functors. Note that $\bar{F} := F \times^{{}^LG} \mathrm{Gal}(E/K)$ is the cyclotomic étale φ-module with $\mathrm{Gal}(E/K)$-structure that comes from the quotient map $\mathrm{Gal}_K \to \mathrm{Gal}(E/K)$. Regarding $\mathrm{Gal}(E/K)$ as a constant group scheme and write $\mathcal{O}_{\mathrm{Gal}(E/K)}$ for its coordinate ring. The representation $\mathrm{Gal}_K \to \mathrm{GL}(\mathcal{O}_{\mathrm{Gal}(E/K)})$ is the induction of the trivial character of Gal_E. By the algorithm in the proof of [Lev13, Theorem 2.5.2], $\bar{F} = \operatorname{Spec} \mathbb{A}_{E,\bar{\mathbb{F}}_p}$. If E/K is ramified, $\bar{F}$ is a non-split $\mathrm{Gal}(E/K)$-torsor. Let $\operatorname{Spec} B \to \operatorname{Spec} \bar{\mathbb{F}}_p$ be a finite type morphism and choose a point $x : \operatorname{Spec} \bar{\mathbb{F}}_p \to \operatorname{Spec} B$. By Nullstellensatz, the composition $\operatorname{Spec} \bar{\mathbb{F}}_p \to \operatorname{Spec} B \to \operatorname{Spec} \bar{\mathbb{F}}_p$ is the identity morphism, and thus F does not split over $\operatorname{Spec} \mathbb{A}_{K,B}$ (if it does, then $\bar{F}$ splits over $\operatorname{Spec} \mathbb{A}_{K,\bar{\mathbb{F}}_p}$, which is a contradiction).

5.2. Rigidifiable étale φ-modules with ${}^LG$-structure

5.2.1. We first define an action

$$m : \mathrm{Gal}(E/K) \times \mathcal{R}_{E,\widehat{G}} \to \mathcal{R}_{E,\widehat{G}},$$

as follows. Let $(F^\circ, \phi_{F^\circ})$ be an object of $\mathcal{R}_{E,\widehat{G}}(A)$. Set $F := F^\circ \times^{\widehat{G}} {}^LG$, which is a ${}^LG$-torsor over $\mathbb{A}_{E,A}$. Choose $\sigma \in \mathrm{Gal}(E/K)$. Note that $\sigma(F^\circ) \subset F$ is a $\widehat{G}$-torsor over $\mathbb{A}_{E,A}$. Set $\sigma \cdot (F^\circ, \phi_{F^\circ}) := (\sigma(F^\circ), \sigma_*\phi_{F^\circ})$, where $\sigma_*\phi_{F^\circ}$ is the restriction of $\phi_F : \varphi^*F \to F$ to $\varphi^*(\sigma(F^\circ)) = \sigma(\varphi^*F^\circ)$. We remark that $\sigma : F^\circ \to \sigma(F^\circ)$, $x \mapsto \sigma(x)$ is a scheme isomorphism but not a $\widehat{G}$-torsor morphism in general.

Assume Theorem 3.7.3 holds. Denote by $\mathcal{R}_{E,{}^LG,m,+}$ the scheme-theoretic image of $\mathrm{Gal}(E/K) \times \mathcal{R}_{E,\widehat{G},m}$ in $\mathcal{R}_{E,\widehat{G}}$. By the universal property of scheme-theoretic image, $\mathrm{Gal}(E/K)$ acts on $\mathcal{R}_{E,{}^LG,m,+}$. So we have the following Lemma.

5.2.2. Lemma Under the assumptions of Theorem 3.7.3, we have an Ind-algebraic presentation

$$[\mathcal{R}_{E,\widehat{G}}/\operatorname{Gal}(E/K)] \cong \varinjlim_m [\mathcal{R}_{E,{}^LG,m,+}/\operatorname{Gal}(E/K)].$$

Proof. It follows from Theorem 3.7.3 and the definitions. □

5.2.3. Definition An *étale φ-module with ${}^LG$-structure and A-coefficients* is an object

$$F \in [{}^f\mathrm{Rep}_{{}^LG}, \mathrm{Mod}_\varphi^{\text{prét}}(A)]^\otimes.$$

Morphisms of étale φ-module with $\widehat{G}$-structure are morphisms in $[{}^f\mathrm{Rep}_{{}^LG}, \mathrm{Mod}_\varphi^{\text{prét}}(A)]^\otimes$. For each ring map $A \to B$, write F_B for $F \otimes_{\mathbb{A}_A} \mathbb{A}_B$ where $\mathbb{A}_A$ is either $\mathbb{A}_{K,A}$ or $\mathbb{A}_{E,A}$ depending on the context.

We say F is *rigidifiable* if there exists an fppf cover $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $F_B = (F^\circ) \times^{\widehat{G}} {}^LG$ for some étale φ-module with $\widehat{G}$-structure F°.

5.2.4. Definition In the following table, we will define the following limit-preserving fppf stacks over $\mathbb{Z}/p^a$. (Compare with Definition 3.1.5.)

Stack over $\mathbb{Z}/p^a$	Objects over finitely presented $\operatorname{Spec} A$
$\mathcal{Z}_{E,{}^LG}$	the groupoid of étale φ-module with ${}^LG$-structure and A-coefficients
$\mathcal{R}_{E,{}^LG}$	the groupoid of rigidifiable étale φ-module with ${}^LG$-structure and A-coefficients.

5.2.5. Lemma (1) The morphism

$$\mathcal{R}_{E,\widehat{G}} \to \mathcal{R}_{E,{}^LG}$$

sending F to $F \times^{\widehat{G}} {}^LG$ factors through the quotient stack $[\mathcal{R}_{E,\widehat{G}}/\operatorname{Gal}(E/K)]$.

(2) The diagonal of $\mathcal{R}_{E,{}^LG}$ and $\mathcal{Z}_{E,{}^LG}$ are representable by algebraic spaces, affine and of finite presentation over $\mathbb{Z}/p^a$.

(3) The canonical morphism

$$[\mathcal{R}_{E,\widehat{G}}/\operatorname{Gal}(E/K)] \to \mathcal{R}_{E,{}^LG}$$

is an equivalence of 2-categories.

Proof. (1) It is immediate from the definition of $\operatorname{Gal}(E/K)$-action explained in Paragraph 5.2.1.

(2) It follows from part (2) of Theorem 3.3.10.

(3) It follows from the definition of quotient stacks. Alternatively, we can use Corollary 2.7.4 if we don't want to unravel the definitions. Choose an embedding ${}^LG \to \mathrm{GL}_N$. Set

$$\mathcal{R}_{E,{}^LG,m} := \mathcal{R}_{E,{}^LG} \underset{\mathcal{R}_{E,\mathrm{GL}_N}}{\times} \mathcal{R}_{E,\mathrm{GL}_N,m}.$$

It is clear that

$$\mathcal{R}_{E,{}^LG,m} \underset{\mathcal{R}_{E,{}^LG}}{\times} \mathcal{R}_{E,\widehat{G}} = \mathcal{R}_{E,{}^LG,m,+}.$$

Repeating the proof of Proposition 3.7.4, we see that $[\mathcal{R}_{E,\widehat{G}}/\operatorname{Gal}(E/K)] \cong \mathcal{R}_{E,{}^LG}$. □

5.2.6. Definition Define the following limit-preserving fppf stacks over $\mathbb{Z}/p^a$:

Prestack over $\mathbb{Z}/p^a$	Objects over finitely presented $\operatorname{Spec} A$
$\mathcal{Z}_{K,{}^LG}$	the groupoid of étale φ-module with ${}^LG$-structure over $\mathbb{A}_{E,A}$
$\mathcal{R}_{K,{}^LG}$	$(\mathcal{R}_{E,{}^LG} \times_{\mathcal{Z}_{E,{}^LG}} \mathcal{Z}_{K,{}^LG})(A)$

Note that $\mathcal{R}_{K,{}^LG} = \mathcal{R}_{E,{}^LG} \times_{\mathcal{Z}_{E,{}^LG}} \mathcal{Z}_{K,{}^LG}$.

5.2.7. Definition An étale φ-module with ${}^LG$-structure and A-coefficients is said to be *E-admissible* if it is an object of $\mathcal{R}_{K,{}^LG}(A)$.

5.2.8. Cyclic Galois descent

5.2.9. The second diagonal of moduli stacks Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Denote by $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, $x \mapsto (x, x)$ the first diagonal of f, and denote by

$$\Delta_{f,2} : \mathcal{X} \to \mathcal{X} \underset{\Delta_f, \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, \Delta_f}{\times} \mathcal{X}$$

the diagonal of the diagonal of f. For each $\operatorname{Spec} A$, the objects of $\mathcal{X} \underset{\Delta_f, \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, \Delta_f}{\times} \mathcal{X}(A)$ are tuples $((x, y), (\alpha, \beta))$ where $x, y \in \mathcal{X}(A)$ and $\alpha, \beta : x \to y$ are isomorphisms; morphisms $((x, y), (\alpha, \beta)) \to ((x', y'), (\alpha', \beta'))$ in $\mathcal{X} \underset{\Delta_f, \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, \Delta_f}{\times} \mathcal{X}(A)$ are pairs $(f, g) : (x, y) \to (x', y')$ fitting in a commutative diagram

$$\begin{tikzcd}
(x,x) \arrow[r, "(\alpha,\beta)"] \arrow[d, "(f,f)"] & (y,y) \arrow[d, "(g,g)"] \\
(x',x') \arrow[r, "(\alpha',\beta')"] & (y',y')
\end{tikzcd}$$

It is easy to see that $((x, x), (1, \alpha^{-1}\beta))$ is isomorphic to $((x, y), (\alpha, \beta))$ for any $((x, y), (\alpha, \beta))$. Therefore $\mathcal{X} \underset{\Delta_f, \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, \Delta_f}{\times} \mathcal{X}(A)$ is equivalent to the groupoid of pairs $(x, \alpha^{-1}\beta)$ where $\alpha^{-1}\beta \in \operatorname{Aut}_{\mathcal{X}}(x)$. The second diagonal $\Delta_{f,2}$ has the following moduli interpretation

$$\begin{aligned} \Delta_{f,2} : \mathcal{X} &\to \mathcal{X} \underset{\Delta_f, \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, \Delta_f}{\times} \mathcal{X} \\ x &\mapsto (x, 1_x). \end{aligned}$$

5.2.10. Lemma If the first and the second diagonal of $\mathcal{R}_{E,{}^LG}$ are both representable by algebraic spaces, affine and of finite presentation, then so is the morphism

$$\begin{aligned} \mathcal{R}_{K,{}^LG} &\to \mathcal{R}_{E,{}^LG} \\ F &\mapsto F \underset{\operatorname{Spec} \mathbb{A}_K/p^a}{\times} \operatorname{Spec} \mathbb{A}_E/p^a. \end{aligned}$$

Proof. Since E/K is a successive extension of cyclic Galois extensions, it is harmless to assume E/K is a cyclic extension with Galois group $\{1, \sigma, \dots, \sigma^{s-1}\}$. By Galois descent (Lemma 4.2.1 and [Stacks, 0CDQ]) and Tannakian formalism (Lemma 1.7), since $\varphi : \mathbb{A}_E \to \mathbb{A}_E$ commutes with the $\operatorname{Gal}(E/K)$ action on $\mathbb{A}_E$, the category $\mathcal{R}_{K,{}^LG}(A)$ is equivalent to the category of pairs (F, φ_σ) where $F \in \mathcal{R}_{E,{}^LG}(A)$ and $\varphi_\sigma : F \to \sigma^*F$ is an isomorphism in the category of étale φ-modules with ${}^LG$-structure, such that the composition

$$F \xrightarrow{\varphi_\sigma} \sigma^*F \xrightarrow{\sigma^*\varphi_\sigma} (\sigma^*)^2F \to \cdots \to (\sigma^*)^sF = F$$

is the identity morphism.

Denote by $\Delta : \mathcal{R}_{E,{}^LG} \to \mathcal{R}_{E,{}^LG} \times \mathcal{R}_{E,{}^LG}$ the diagonal morphism $F \mapsto (F, F)$, and denote by Γ_σ the morphism $\Delta : \mathcal{R}_{E,{}^LG} \to \mathcal{R}_{E,{}^LG} \times \mathcal{R}_{E,{}^LG}$ the diagonal morphism $F \mapsto (F, \sigma^*F)$. The following fiber product

$$\mathcal{Z} := \mathcal{R}_{E,{}^LG} \underset{\Delta, \mathcal{R}_{E,{}^LG}\times\mathcal{R}_{E,{}^LG}, \Gamma_\sigma}{\times} \mathcal{R}_{E,{}^LG}$$

represents the groupoid of pairs (F, φ_σ) such that $\varphi_\sigma : F \to \sigma^*F$ is an isomorphism by unravelling the definitions of 2-fiber product.

Define the following morphism

$$c : \mathcal{Z} \to \mathcal{R}_{E,{}^LG} \underset{\Delta, \mathcal{R}_{E,{}^LG} \times \mathcal{R}_{E,{}^LG}, \Delta}{\times} \mathcal{R}_{E,{}^LG}$$

$$(F, \varphi_\sigma) \mapsto (F, \varphi_\sigma \circ \sigma^* \varphi_\sigma \circ \dots (\sigma^*)^{s-1} \varphi_\sigma)$$

By the discussion in paragraph 5.2.9, we see that $\mathcal{R}_{K,{}^LG}$ is isomorphic to the pullback of $\mathcal{Z}$ along the second diagonal of $\mathcal{R}_{E,{}^LG}$. Let $X \to \mathcal{R}_{E,{}^LG}$ be a morphism whose target is an affine scheme. We have

$$\mathcal{Z} \times_{\mathcal{R}_{E,{}^LG}} X = \mathcal{R}_{E,{}^LG} \underset{\Delta, \mathcal{R}_{E,{}^LG} \times \mathcal{R}_{E,{}^LG}, \Gamma_\sigma}{\times} X$$

which is an affine scheme of finite presentation over X. We have

$$\mathcal{R}_{K,{}^LG} \times_{\mathcal{R}_{E,{}^LG}} X = (\mathcal{Z} \times_{\mathcal{R}_{E,{}^LG}} X) \underset{c,\ \mathcal{R}_{E,{}^LG} \underset{\Delta, \mathcal{R}_{E,{}^LG} \times \mathcal{R}_{E,{}^LG}, \Delta}{\times} \mathcal{R}_{E,{}^LG},\ \Delta_\Delta}{\times} \mathcal{R}_{E,{}^LG}$$

which is an affine scheme of finite presentation over X. ☐

5.3. The construction of $\mathcal{R}_{K,{}^LG,\widehat{G}}$

5.3.1. Definition The stack $\mathcal{R}_{K,{}^LG,\widehat{G}}$ is defined to be

$$\mathcal{R}_{K,{}^LG} \underset{\mathcal{R}_{E,{}^LG}}{\times} \mathcal{R}_{E,\widehat{G}}.$$

5.3.2. Theorem If $\varphi : \mathbb{A}_E/p^a \to \mathbb{A}_E/p^a$ admits a height theory (Definition 2.6.1), $\mathcal{R}_{K,{}^LG}$ and $\mathcal{R}_{K,{}^LG,\widehat{G}}$ are representable by limit-preserving Ind-algebraic stacks over $\mathbb{Z}/p^a$, whose diagonal is representable by algebraic spaces, affine and of finite presentation.

Proof. It follows from Theorem 3.7.1, Lemma 5.2.2, Lemma 5.2.10 and [EG23, Corollary 3.2.9]. When applying Lemma 5.2.10, we need the affine representability of the second diagonal, which follows from [Stacks, 04YZ]. ☐

5.3.3. Corollary Assume either $a = 1$ or $E/\mathbb{Q}_p$ is a tame extension. After possibly replacing E by an unramified extension, $\mathcal{R}_{K,{}^LG}$ and $\mathcal{R}_{K,{}^LG,\widehat{G}}$ are representable by limit-preserving Ind-algebraic stacks over $\mathbb{Z}/p^a$, whose diagonals are representable by algebraic spaces, affine and of finite presentation.

Proof. It follows from Theorem 5.3.2 and Lemma 4.2.2. ☐

6. Step 3: the moduli of cyclotomic étale (φ, Γ)-modules

In this section, E/K is a finite, tamely ramified, Galois extension.
See Definition 4.2.4 for the definition of (φ, Γ)-modules.
We often drop the subscript "K" from $\mathcal{R}_{K,{}^LG}$ and write $\mathcal{R}_{{}^LG} = \mathcal{R}_{K,{}^LG}$.

6.1. Étale (φ, Γ)-modules with ${}^LG$-structure

6.1.1. Definition Let A be an $\mathbb{Z}/p^a$-algebra. We omit the phrase "with A-coefficients" in the following definitions.

- An *étale* (φ, γ)*-module with* ${}^LG$*-structure* is a faithful, exact, symmetric monoidal functor from ${}^f\operatorname{Rep}_G$ to the category of étale (φ, γ)-modules with A-coefficients.
- An *étale* (φ, Γ)*-module with* ${}^LG$*-structure* is a faithful, exact, symmetric monoidal functor from ${}^f\operatorname{Rep}_G$ to the category of étale (φ, Γ)-modules with A-coefficients.

See Definition 5.2.7 for the definition of E-admissibility.

We also define the following limit preserving fppf stacks over $\mathbb{Z}/p^a$:

Stack	Objects over finitely presented $\operatorname{Spec} A$
$\mathcal{Z}^{\gamma}_{{}^LG}$	the groupoid of étale (φ, γ)-module with ${}^LG$-structure and A-coefficients
$\mathcal{Z}^{\Gamma}_{{}^LG}$	the groupoid of étale (φ, Γ)-module with ${}^LG$-structure and A-coefficients
$\mathcal{R}^{\gamma}_{{}^LG}$	the groupoid of E-admissible étale (φ, γ)-module with ${}^LG$-structure and A-coefficients
$\mathcal{R}^{\Gamma}_{{}^LG}$	the groupoid of E-admissible étale (φ, Γ)-module with ${}^LG$-structure and A-coefficients
$\mathcal{X}_d^{\Gamma_{\mathrm{disc}}}$	the groupoid of projective étale (φ, γ)-module of rank d with A-coefficients
$\mathcal{X}_d$	the groupoid of projective étale (φ, Γ)-module of rank d with A-coefficients

Moreover, we also define the following fiber products:

$$\mathcal{R}^{\Gamma}_{K,{}^LG,\widehat{G}} = \mathcal{R}^{\Gamma}_{{}^LG,\widehat{G}} := \mathcal{R}^{\Gamma}_{{}^LG} \underset{\mathcal{R}_{E,{}^LG}}{\times} \mathcal{R}_{E,\widehat{G}},$$
$$\mathcal{Z}^{\Gamma}_{K,{}^LG,\widehat{G}} = \mathcal{Z}^{\Gamma}_{{}^LG,\widehat{G}} := \mathcal{Z}^{\Gamma}_{{}^LG} \underset{\mathcal{Z}_{E,{}^LG}}{\times} \mathcal{Z}_{E,\widehat{G}}.$$

6.1.2. Lemma Let ${}^LG \to \mathrm{GL}_d$ be a faithful algebraic representation. The following diagram of stacks over $\mathbb{Z}/p^a$ is Cartesian:

$$\begin{tikzcd} \mathcal{Z}^{\Gamma}_{{}^LG} \arrow[r] \arrow[d] & \mathcal{X}_d \arrow[d] \\ \mathcal{Z}^{\gamma}_{{}^LG} \arrow[r] & \mathcal{X}_d^{\Gamma_{\mathrm{disc}}} \end{tikzcd}$$

In particular, $\mathcal{Z}^{\Gamma}_{{}^LG} \to \mathcal{Z}^{\gamma}_{{}^LG}$ is a monomorphism.

Proof. By [Stacks, 07XM], the fiber product of limit preserving stacks is limit preserving. Note that both stacks on the right are limit-preserving by [EG23, Lemma 3.2.19 and Lemma 3.2.15].

So it suffices to check it is a Cartesian diagram when restricted to finitely presented test objects. By [Lev13, Theorem C.1.7], ${}^f\operatorname{Rep}_G$ is generated as a tensor category by ${}^LG \to \mathrm{GL}_d$. It remains to show the topological nilpotency of γ-structure of étale (φ, γ)-modules is preserved by taking tensor products and subquotients. It follows immediately from [EG23, Lemma D.21, Lemma D.27], with the caveat that that $\mathbb{A}_{K,A}$ does not satisfy the assumptions of [EG23, Definition D.17]. By [EG23, Lemma 3.2.16], the topological nilpotency can be checked by regarding $\mathbb{A}_{K,A}$-modules as $\mathbb{A}_{K^{\mathrm{basic}},A}$-modules and $\mathbb{A}_{K^{\mathrm{basic}},A}$ satisfies the assumptions of the assumptions of [EG23, Definition D.17] (see loc. cit. for the undefined notation). ◻

6.1.3. Lemma The following diagrams are Cartesian.

$$\begin{tikzcd} \mathcal{R}^{\gamma}_{{}^LG} \arrow[r] \arrow[d] & \mathcal{R}_{K,{}^LG} \arrow[d] \\ \mathcal{Z}^{\gamma}_{{}^LG} \arrow[r] & \mathcal{Z}_{K,{}^LG} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}^{\Gamma}_{{}^LG} \arrow[r] \arrow[d] & \mathcal{R}_{K,{}^LG} \arrow[d] \\ \mathcal{Z}^{\Gamma}_{{}^LG} \arrow[r] & \mathcal{Z}_{K,{}^LG} \end{tikzcd}$$

Proof. By [Stacks, 07XM], the fiber product of limit preserving stacks is limit preserving. So it is harmless to restrict to finitely presented test objects; but then it is how the stacks on the top left corners are defined. □

Since the φ-structure and γ-structure on $\mathbb{A}_{K,A}$ commute, for any étale φ-module M, we can define an étale φ-module $\gamma(M) := \mathbb{A}_{K,A} \otimes_{\gamma,\mathbb{A}_{K,A}} M$. By Tannakian formalism, for any any étale φ-module F with G-structure, we can define $\gamma(F)$; and thus we defined a canonical morphism $\gamma : \mathcal{R}_{{}^LG} \to \mathcal{R}_{{}^LG}$.

6.1.4. Theorem Assume either $a = 1$ or $E/\mathbb{Q}_p$ is a tame extension.

(1) The stack $\mathcal{R}^{\gamma}_{{}^LG}$ is represented by the 2-fiber product

$$\mathcal{R}_{K,{}^LG} \underset{\Delta,\mathcal{R}_{K,{}^LG}\times\mathcal{R}_{K,{}^LG},\Gamma_\gamma}{\times} \mathcal{R}_{K,{}^LG}$$

where $\Delta : x \mapsto (x, x)$ is the diagonal map, and $\Gamma_\gamma : x \mapsto (x, \gamma(x))$ is the graph of γ.

(2) $\mathcal{R}^{\gamma}_{{}^LG}$ is a limit preserving Ind-algebraic stack over $\mathbb{Z}/p^a$ whose diagonal is representable by algebraic spaces, affine, and of finite presentation.

(3) The diagonal of $\mathcal{Z}^{\Gamma}_{{}^LG}$ and $\mathcal{R}^{\Gamma}_{{}^LG}$ are representable by algebraic spaces, affine, and of finite presentation.

(4) The stack $\mathcal{R}^{\Gamma}_{{}^LG}$ is an Ind-algebraic stack over $\mathbb{Z}/p^a$, and is an inductive limit of algebraic stacks of finite type over $\mathbb{Z}/p^a$, with transition morphisms that are closed immersions.

Proof. (1) Similar to [EG23, Proposition 3.2.14].

(2) By Corollary 5.3.3, the diagonal map $\Delta : \mathcal{R}_{K,{}^LG} \to \mathcal{R}_{K,{}^LG} \times_{\mathbb{Z}/p^a} \mathcal{R}_{K,{}^LG}$ is representable by algebraic spaces, affine, and of finite presentation. By (1), the forgetful morphism $\mathcal{R}^{\gamma}_{{}^LG} \to \mathcal{R}_{K,{}^LG}$ is a base change of the diagonal morphism $\Delta : \mathcal{R}_{K,{}^LG} \to \mathcal{R}_{K,{}^LG} \times_{\mathbb{Z}/p^a} \mathcal{R}_{K,{}^LG}$, and thus by [EG23, Corollary 3.2.9] is representable by algebraic spaces, affine, and of finite presentation.

(3) It follows from (2) and that $\mathcal{R}^{\Gamma}_{G} \to \mathcal{R}^{\gamma}_{G}$ is a monomorphism.

(4) Note that both $\mathcal{X}_d$ and $\mathcal{R}^{\gamma}_{G}$ are Ind-algebraic stacks that are the inductive limit of algebraic stacks of finite presentation over $\mathbb{Z}/p^a$. Moreover, the inductive limit structure on $\mathcal{X}_d$ and $\mathcal{R}^{\gamma}_{G}$ are compatible (see Paragraph 6.1.5). Part (4) thus follows from part (2), Lemma 6.1.2 and Lemma 6.1.3. □

6.1.5. The Ind-algebraic presentation of $\mathcal{R}_{{}^LG^\Gamma}$ By Lemma 5.2.2 and Lemma 5.2.5, there is an Ind-algebraic presentation

$$\mathcal{R}_{E,{}^LG} \cong \varinjlim_m [\mathcal{R}_{E,{}^LG,m,+}/\operatorname{Gal}(E/\mathbb{Q}_p)].$$

Set

$$\begin{aligned}\mathcal{R}_{{}^LG,m} &:= [\mathcal{R}_{E,m,{}^LG,+}/\operatorname{Gal}(E/\mathbb{Q}_p)] \times_{\mathcal{R}_{E,{}^LG}} \mathcal{R}_{K,{}^LG},\\ \mathcal{R}^{\gamma}_{{}^LG,m} &:= \mathcal{R}_{{}^LG,m} \times_{\mathcal{R}_{{}^LG}} \mathcal{R}^{\gamma}_{{}^LG},\\ \mathcal{R}^{\Gamma}_{{}^LG,m} &:= \mathcal{R}^{\gamma}_{{}^LG,m} \times_{\mathcal{R}^{\gamma}_{{}^LG}} \mathcal{R}^{\Gamma}_{{}^LG}.\end{aligned}$$

We have $\mathcal{R}^{\Gamma}_{{}^LG} = \varinjlim_m \mathcal{R}^{\Gamma}_{{}^LG,m}$.

6.2. The connection with Galois representations

In this subsection, we relate the stack $\mathcal{R}^{\Gamma}_{{}^LG}$ to Galois representations.

6.2.1. Definition Let $\mathfrak{P}$ be a profinite group and let A be an Artinian $\mathbb{Z}_p$-algebra. Denote by $\mathcal{R}_{\mathfrak{P},{}^LG}(A)$ the groupoid consisting of pairs $(\mathcal{T},\rho)$ where $\mathcal{T}$ is an ${}^LG$-torsor over $\operatorname{Spec} A$, and $\rho : \mathfrak{P} \to \operatorname{Aut}_{{}^LG}(\mathcal{T})$ is a continuous homomorphism from $\mathfrak{P}$ to the group of torsor automorphisms of $\mathcal{T}$. Morphisms $(\mathcal{T}_1,\rho_1) \to (\mathcal{T}_2,\rho_2)$ are ${}^LG$-torsor morphisms $f : \mathcal{T}_1 \to \mathcal{T}_2$ such that $\rho_2 = f_*\rho_1$.

6.2.2. Lemma Let A be an Artinian $\mathbb{Z}/p^a$-algebra. The groupoid $\mathcal{R}_{E,{}^LG}(A)$ is equivalent to the essential image of

$$\mathcal{R}_{\operatorname{Gal}_{E_\infty},\widehat{G}}(A) \xrightarrow{(\mathcal{T},\rho)\mapsto(\mathcal{T}\times^{\widehat{G}}{}^LG,\rho)} \mathcal{R}_{\operatorname{Gal}_{E_\infty},{}^LG}(A).$$

Proof. It follows from Tannakian formalism and Definition 5.2.6. Note that $\widehat{G}$-torsors over $\mathbb{A}_{E,A}$ are automatically rigidifiable by Lemma 2.7.7. ☐

6.2.3. Lemma Let A be an Artinian $\mathbb{Z}/p^a$-algebra. The groupoid $\mathcal{R}_{K,{}^LG}(A)$ is equivalent to the full subcategory of $\mathcal{R}_{\operatorname{Gal}_{K_\infty},{}^LG}(A)$ consisting of objects which are sent to the essential image of

$$\mathcal{R}_{\operatorname{Gal}_{E_\infty},\widehat{G}}(A) \xrightarrow{(\mathcal{T},\rho)\mapsto(\mathcal{T}\times^{\widehat{G}}{}^LG,\rho)} \mathcal{R}_{\operatorname{Gal}_{E_\infty},{}^LG}(A)$$

under the morphism $\mathcal{R}_{\operatorname{Gal}_{K_\infty},{}^LG} \to \mathcal{R}_{\operatorname{Gal}_{E_\infty},{}^LG}$.

Proof. It follows from Tannakian formalism and Definition 5.2.6. ☐

6.2.4. Lemma Let A be an Artinian $\mathbb{Z}/p^a$-algebra. The groupoid $\mathcal{R}_{{}^LG}(A)$ is equivalent to the full subcategory of $\mathcal{R}_{\operatorname{Gal}_K,{}^LG}(A)$ consisting of objects which are sent to the essential image of

$$\mathcal{R}_{\operatorname{Gal}_{E_\infty},\widehat{G}}(A) \xrightarrow{(\mathcal{T},\rho)\mapsto(\mathcal{T}\times^{\widehat{G}}{}^LG,\rho)} \mathcal{R}_{\operatorname{Gal}_{E_\infty},{}^LG}(A).$$

under the morphism $\mathcal{R}_{\operatorname{Gal}_K,{}^LG} \to \mathcal{R}_{\operatorname{Gal}_{E_\infty},{}^LG}$.

Proof. It follows from Tannakian formalism and the definitions. ☐

6.2.5. Definition Let $\mathfrak{P}$ be a profinite group and let A be an Artinian local $\mathbb{Z}_p$-algebra. Denote by $\mathcal{R}_{\mathfrak{P},{}^LG,\widehat{G}}(A)$ the groupoid consisting of tuples $(\mathcal{T},\rho,c)$ where $(\mathcal{T},\rho) \in \mathcal{R}_{\mathfrak{P},{}^LG}(A)$ and c is a connected component of the scheme $\mathcal{T} \otimes_A \bar{\mathbb{F}}_p$. Morphisms of $(\mathcal{T}_1,\rho_1,c_1) \to (\mathcal{T}_2,\rho_2,c_2)$ are morphisms $(\mathcal{T}_1,\rho_1) \to (\mathcal{T}_2,\rho_2)$ that send c_1 to c_2.

6.2.6. Lemma Let A be an Artinian local $\mathbb{Z}_p$-algebra. The groupoid $\mathcal{R}_{K,{}^LG,\widehat{G}}(A)$ is equivalent to the full subcategory of the groupoid $\mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG,\widehat{G}}(A)$ consisting of objects that are sent to the essential image

$$\mathcal{R}_{\mathrm{Gal}_{E_\infty},\widehat{G}}(A) \xrightarrow{(\mathcal{T},\rho)\mapsto(\mathcal{T}\times^{\widehat{G}}{}^LG,\rho)} \mathcal{R}_{\mathrm{Gal}_{E_\infty},{}^LG}(A).$$

under the morphism $\mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG,\widehat{G}} \to \mathcal{R}_{\mathrm{Gal}_{E_\infty},{}^LG}$.

Proof. Since

$$\mathcal{R}_{K,{}^LG,\widehat{G}} = \mathcal{R}_{K,{}^LG} \underset{\mathcal{R}_{E,{}^LG}}{\times} \mathcal{R}_{E,\widehat{G}},$$

an object of $\mathcal{R}_{K,{}^LG,\widehat{G}}$ can be described as a pair (F, F°) where F is an étale φ-module with ${}^LG$-structure over $\mathbb{A}_{K,A}$ and F° is an étale φ-module with $\widehat{G}$-structure over $\mathbb{A}_{E,A}$, together with an identification

$$F \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A} = (F^\circ) \times^{\widehat{G}} {}^LG.$$

F is sent to an object $(\mathcal{T}, \rho)$ of $\mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG}(A)$; and F° is sent to an object $(\mathcal{T}^\circ, \rho_E)$ of $\mathcal{R}_{\mathrm{Gal}_{E_\infty},{}^LG}(A)$. We also have an identification $\mathcal{T} = \mathcal{T}^\circ \times^{\widehat{G}} {}^LG$. In particular, $\mathcal{T}^\circ$ determines a connected component c of the $\mathcal{T}$. So we get a functor $(F, F^\circ) \mapsto (\mathcal{T}, \rho, c)$ from $\mathcal{R}_{K,{}^LG,\widehat{G}}(\bar{\mathbb{F}}_p)$ to $\mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG,\widehat{G}}(\bar{\mathbb{F}}_p)$. The essential image of the functor is clearly what we described in the lemma. So it remains to show the functor is fully faithful.

Morphisms $(F_1, F_1^\circ) \to (F_2, F_2^\circ)$ are pairs $(\alpha : F_1 \to F_2, \beta : F_1^\circ \to F_2^\circ)$ such that $(*)$ $\alpha \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A} = \beta \times^{\widehat{G}} {}^LG$. By working étale locally, we can assume $F \cong \operatorname{Spec} \mathbb{A}_{K,A} \times {}^LG$ and $F^\circ \cong \operatorname{Spec} \mathbb{A}_{E,A} \times \widehat{G}$ are trivial torsors. So α is represented by a matrix $\alpha \in {}^LG(\mathbb{A}_{K,A})$ and β is represented by a matrix $\beta \in \widehat{G}(\mathbb{A}_{E,A})$. The equation $(*)$ becomes simply $\alpha = \beta$. So the set of morphisms are contained in ${}^LG(\mathbb{A}_{K,A}) \cap \widehat{G}(\mathbb{A}_{E,A}) = \widehat{G}(\mathbb{A}_{K,A})$. The forgetful morphisms $\mathcal{R}_{K,{}^LG,\widehat{G}} \to \mathcal{R}_{K,{}^LG}$ and $\mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG,\widehat{G}} \to \mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG}$ are both faithful, and thus $\mathcal{R}_{K,{}^LG,\widehat{G}} \to \mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG,\widehat{G}}$ is faithful. It follows from the description above that the functor $\mathcal{R}_{K,{}^LG,\widehat{G}} \to \mathcal{R}_{\mathrm{Gal}_{K_\infty},{}^LG,\widehat{G}}$ induces bijections between morphisms. $\square$

6.2.7. Theorem let A be an Artinian local $\mathbb{Z}_p$-algebra. The groupoid $\mathcal{R}^{\Gamma}_{{}^LG,\widehat{G}}(A)$ is equivalent to the full subcategory of the groupoid $\mathcal{R}_{\mathrm{Gal}_K,{}^LG,\widehat{G}}(A)$ consisting of objects that are sent to the essential image

$$\mathcal{R}_{\mathrm{Gal}_{E_\infty},\widehat{G}}(A) \xrightarrow{(\mathcal{T},\rho)\mapsto(\mathcal{T}\times^{\widehat{G}}{}^LG,\rho)} \mathcal{R}_{\mathrm{Gal}_{E_\infty},{}^LG}(A).$$

under the morphism $\mathcal{R}_{\mathrm{Gal}_K,{}^LG,\widehat{G}} \to \mathcal{R}_{\mathrm{Gal}_{E_\infty},{}^LG}$.

Proof. It follows from Lemma 6.2.4 and Lemma 6.2.6. $\square$

6.2.8. Remark If $\rho : \mathrm{Gal}_K \to {}^LG = \widehat{G} \rtimes \mathrm{Gal}(E/K)$ is an L-parameter, then $\rho(\mathrm{Gal}_E) \subset \widehat{G}$. So $\rho|_{\mathrm{Gal}_{E_\infty}}$ lies in the essential image of

$$\mathcal{R}_{\mathrm{Gal}_{E_\infty},\widehat{G}}(A) \xrightarrow{(\mathcal{T},\rho)\mapsto(\mathcal{T}\times^{\widehat{G}}{}^LG,\rho)} \mathcal{R}_{\mathrm{Gal}_{E_\infty},{}^LG}(A).$$

7. Step 4: the moduli of L-parameters and the restriction morphism

In this section, we define the moduli stack $\mathcal{X}_{{}^LG}$ of L-parameters for G.

7.0.1. The stack $\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}}$

Let $\{*\}$ be the the group with only one element. The L-group of $\{*\}$ is

$$ {}^L\{*\} = \{*\} \rtimes \mathrm{Gal}(E/K) \cong \mathrm{Gal}(E/K). $$

By Theorem 6.2.7, the groupoid $\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}}(\mathbb{Z}/p^a)$ is the groupoid of continuous group homomorphisms $\rho : \mathrm{Gal}_K \to \mathrm{Gal}(E/K)$ such that $\rho|_{\mathrm{Gal}_{E_\infty}}$ is is the trivial group homomorphism. Since $\mathrm{Gal}(E_\infty/E)$ is a pro-p group and $\mathrm{Gal}(E/K)$ has prime-to-p cardinality, $\rho|_{\mathrm{Gal}_E}$ must be the trivial group homomorphism. Therefore

$$ \mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}}(\mathbb{Z}/p^a) = \{\text{Group homomorphisms } \mathrm{Gal}(E/K) \to \mathrm{Gal}(E/K)\}. $$

Note that there is a distinguished element $\mathrm{id} \in \mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}}(\mathbb{Z}/p^a)$ which corresponds to the identity homomorphism.

7.0.2. Lemma The morphism $\mathrm{id} : \mathrm{Spec}\,\mathbb{Z}/p^a \to \mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}}$ is an open and closed immersion of schemes over $\mathbb{Z}/p^a$.

Proof. Since the Ind-algebraic stack $\mathcal{R}_{E,\{*\}}$ has only one $\bar{\mathbb{F}}_p$-point and is a setoid, we conclude that $\mathcal{R}_{E,\{*\}} = \varinjlim_m \mathrm{Spec}\, S_m$ where each S_m is an Artinian local $\mathbb{Z}/p^a$-algebra. In particular, $\mathcal{R}_{E,\{*\}}$ is a subgroupoid of its completion at the unique closed point. By Galois deformation theory (Lemma 6.2.2), $\varinjlim_m \mathrm{Spec}\, S_m = \mathrm{Spec}\,\mathbb{Z}/p^a$.

The diagonal of $\mathcal{R}_{E,{}^L\{*\}}$ is a finite morphism. By Lemma 5.2.10, $\mathcal{R}_{K,{}^L\{*\}} \to \mathcal{R}_{E,{}^L\{*\}}$ is representable by a finite morphism. Since $\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}} \cong \mathcal{R}_{K,{}^L\{*\},\{*\}}$ is an $\mathrm{Gal}(E/K)$-torsor over $\mathcal{R}_{K,{}^L\{*\}}$, we see that $\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}} \cong \mathrm{Spec}\, R$ where R is a finite $\mathbb{Z}/p^a$-module. By Theorem 6.2.7 again, $\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}}$ is a disjoint union of copies of $\mathrm{Spec}\,\mathbb{Z}/p^a$. ☐

7.1. The stack $\mathcal{X}_{{}^LG}$

7.1.1. Definition Set

$$ \mathcal{X}_{{}^LG} := \mathcal{R}^{\Gamma}_{{}^LG,\widehat{G}} \underset{\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}},\mathrm{id}}{\times} \mathrm{Spec}\,\mathbb{Z}/p^a, $$

and call it the *Emerton-Gee stack* of L-parameters for G over $\mathbb{Z}/p^a$. We also define

$$ \mathcal{X}_{\mathcal{Z},{}^LG} := \mathcal{Z}^{\Gamma}_{{}^LG,\widehat{G}} \underset{\mathcal{R}^{\Gamma}_{{}^L\{*\},\{*\}},\mathrm{id}}{\times} \mathrm{Spec}\,\mathbb{Z}/p^a. $$

Roughly speaking, $\mathcal{X}_{\mathcal{Z},{}^LG}$ is the largest possible limit-preserving stack that classifies étale (φ,Γ)-modules that algebraically interpolate L-parameters, and $\mathcal{X}_{{}^LG} \subset \mathcal{X}_{\mathcal{Z},{}^LG}$ is the full substack that classifies E-admissible objects.

7.1.2. Theorem Assume either $a = 1$ or $E/\mathbb{Q}_p$ is a tame extension.

(1) The stack $\mathcal{X}_{^LG}$ is limit preserving over $\mathbb{Z}/p^a$.

(2) The diagonal of $\mathcal{X}_{^LG}$ is representable by algebraic spaces, affine, and of finite presentation.

(3) The stack $\mathcal{X}_{^LG}$ is an Ind-algebraic stack over $\operatorname{Spec}\mathbb{Z}/p^a$, and is an inductive limit of algebraic stacks of finite type over $\mathbb{Z}/p^a$, with transition morphisms that are closed immersions.

Proof. It follows from Theorem 6.1.4 and the definition of $\mathcal{X}_{^LG}$. □

7.1.3. Proposition Let A be an Artinian local $W(\bar{\mathbb{F}}_p)$-algebra. There is a bijection between the set of L-parameters $H^1(\mathrm{Gal}_K, \widehat{G}(A))$ with the set of equivalence classes in the groupoid $\mathcal{X}_{^LG}(A)$.

Proof. By Theorem 6.2.7, the groupoid $\mathcal{X}_{^LG}(A)$ is equivalent to the groupoid of tuples $(\mathcal{T}, \rho, c)$ where $\mathcal{T}$ is an ${}^LG$-torsor over $\operatorname{Spec} A$, ρ is a continuous action of Gal_K on $\mathcal{T}$ and c is a connected component of $\mathcal{T}$, such that $(\mathcal{T}, \rho, c) \times^{^LG} {}^L\{*\}$ is the distinguished element id. Since $\bar{\mathbb{F}}_p$ is an algebraically closed field, we can replace $\mathcal{T}$ by the trivial ${}^LG$-torsor ${}^LG$. Since $({}^LG, \rho, c) \times^{^LG} {}^L\{*\} = \mathrm{id}$, c must be the distinguished component $\widehat{G} \subset {}^LG$. ρ is presented by a continuous group homomorphism $\mathrm{Gal}_K \to {}^LG(A)$. A morphism $({}^LG, \rho_1, c) \to ({}^LG, \rho_2, c)$ is a matrix $g \in {}^LG$ such that $\rho_1 = g\rho_2 g^{-1}$ and that $gc = c$, that is, $g \in \widehat{G}(A)$. □

7.2. Geometric non-abelian Shapiro's lemma

Assume $E/\mathbb{Q}_p$ is tame. We prove $\mathcal{X}_{^LG} = \mathcal{X}_{^L\mathrm{Res}_{K/\mathbb{Q}_p} G}$ in this section.

The non-abelian Shapiro's lemma is known for Galois representations.

7.2.1. Lemma (The non-abelian Shapiro's lemma) Let A be an Artinian $\mathbb{Z}_p$-algebra. There is a natural bijection

$$H^1(\mathrm{Gal}_K, \widehat{G}(A)) \cong H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \widehat{\mathrm{Res}_{K/\mathbb{Q}_p}}\, G(A)).$$

Proof. It is a special case of [St10, Proposition 8]. □

7.2.2. A diagonal trick

By [St10, 2.1.5], there exists a diagram of group homomorphisms

$$\begin{tikzcd}
\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}}\, G \rtimes \mathrm{Gal}(E/K) \arrow[r] \arrow[d] & {}^LG \arrow[r, equal] & \widehat{G} \rtimes \mathrm{Gal}(E/K) \\
{}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G) \arrow[r, equal] & \widehat{\mathrm{Res}_{K/\mathbb{Q}_p}}\, G \rtimes \mathrm{Gal}(E/\mathbb{Q}_p) &
\end{tikzcd}$$

Denote by H the group $\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}}\, G \rtimes \mathrm{Gal}(E/K)$.

We define the auxiliary stack

$$\mathcal{Z} := \mathcal{X}_{^L(\mathrm{Res}_{K/\mathbb{Q}_p} G)} \underset{\mathcal{R}^\Gamma_{K, {}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G), (\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}} G)}}{\times} \mathcal{R}^\Gamma_{K, H, \widehat{\mathrm{Res}_{K/\mathbb{Q}_p}} G} \underset{\mathcal{R}^\Gamma_{^LG, \widehat{G}}}{\times} \mathcal{X}_{^LG}.$$

Since $\mathcal{X}_{^LG} \to \mathcal{R}^\Gamma_{^LG, \widehat{G}}$ is an open and closed immersion, the $\operatorname{Spec} A$-objects of $\mathcal{Z}$ are tuples (F, F_K, f) where F is an étale (φ, Γ)-module with ${}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G)$-structure over $\mathbb{A}_{\mathbb{Q}_p, A}$ with additional structures,

F_K is an étale (φ,Γ)-module with H over $\mathbb{A}_{K,A}$ with additional structures, and f is an identification $F\otimes_{\mathbb{A}_{\mathbb{Q}_p,A}}\mathbb{A}_{K,A}\cong F_K\times^{H}{}^{L}(\mathrm{Res}_{K/\mathbb{Q}_p}G)$. We have the forgetful morphisms

$$\begin{aligned}
&\mathcal{Z}\to\mathcal{X}_{^L(\mathrm{Res}_{K/\mathbb{Q}_p}G)}, && (F,F_K,f)\mapsto F;\\
&\mathcal{Z}\to\mathcal{X}_{^LG}, && (F,F_K,f)\mapsto F_K\times^{H}{}^{L}G.
\end{aligned}$$

7.2.3. Lemma If A is an Artinian $W(\bar{\mathbb{F}}_p)$-algebra, then

$$\begin{aligned}
&\mathcal{Z}(A)\to\mathcal{X}_{^L(\mathrm{Res}_{K/\mathbb{Q}_p}G)}(A)\\
&\mathcal{Z}(A)\to\mathcal{X}_{^LG}(A)
\end{aligned}$$

are both equivalence of categories.

Proof. By Proposition 7.1.3, the objects of $\mathcal{X}_{^L\mathrm{Res}_{K/\mathbb{Q}_p}G}(A)$ are continuous homomorphisms $\rho:\mathrm{Gal}_{\mathbb{Q}_p}\to{}^L\mathrm{Res}_{K/\mathbb{Q}_p}G(A)$ that are compatible with the quotient map $\mathrm{Gal}(\mathbb{Q}_p)\to\mathrm{Gal}(E/\mathbb{Q}_p)$. In particular, $\rho|_{\mathrm{Gal}_K}$ factors necessarily through $H(A)$. The lemma now follows from the Shapiro's Lemma 7.2.1. $\square$

7.2.4. Proposition (Geometric Shapiro's Lemma) There exists a canonical isomorphism $\mathrm{Sha}:\mathcal{X}_{^LG}\xrightarrow{\cong}\mathcal{X}_{^L\mathrm{Res}_{K/\mathbb{Q}_p}G}$.

Proof. By Lemma 7.2.3 and Corollary 2.7.4,

$$\begin{aligned}
&\mathcal{Z}\to\mathcal{X}_{^L(\mathrm{Res}_{K/\mathbb{Q}_p}G)}\\
&\mathcal{Z}\to\mathcal{X}_{^LG}
\end{aligned}$$

are both isomorphisms. $\square$

The same argument shows that the construction of $\mathcal{X}_{^LG}$ does not depend on the splitting field E.

7.2.5. Proposition (Independence of the splitting field) Let E, E' be two fields that splits G. We have a canonical isomorphism

$$\mathcal{X}_{\widehat{G}\rtimes\mathrm{Gal}(E/K)}\cong\mathcal{X}_{\widehat{G}\rtimes\mathrm{Gal}(E'/K)}.$$

Proof. Without loss of generality, we may assume E' contains the Galois closure of E. There is a canonical morphism

$$f:\mathcal{X}_{\widehat{G}\rtimes\mathrm{Gal}(E'/K)}\to\mathcal{X}_{\widehat{G}\rtimes\mathrm{Gal}(E/K)}$$

sending F to $F\times^{\widehat{G}\rtimes\mathrm{Gal}(E'/K)}(\widehat{G}\rtimes\mathrm{Gal}(E/K))$. By Corollary 2.7.4, f is an isomorphism. $\square$

7.3. The case of tori

In this subsection, we only need to assume E is tame over K.

For each torus T, the norm map $\mathrm{Res}_{E/K}T_E\to T$ induces an L-homomorphism ${}^LT\to{}^L\mathrm{Res}_{E/K}T_E$, which, in turn, induces the restriction morphism

$$\mathcal{X}_{^LT}\to\mathcal{X}_{^L\mathrm{Res}_{E/K}T_E}\cong\mathcal{X}_{^LT_E}.$$

The functoriality of local Langlands correspondence for tori states that the norm map on the automorphic side corresponds to the restriction map on the spectral side.

7.3.1. Lemma The restriction morphism $\mathcal{X}_{^LT} \to \mathcal{X}_{^LT_E}$ is representable by algebraic spaces, affine, and of finite presentation.

Proof. After unravelling the definitions, the lemma follows from Lemma 5.2.10. □

7.3.2. Lemma For each tame torus T, $\mathcal{X}_{^LT}$ is an algebraic stack of finite presentation over $\mathbb{Z}/p^a$.

Proof. By the independence of splitting field (Proposition 7.2.5), we have $\mathcal{X}_{^LT_E} = \mathcal{X}_{\mathrm{G}_m^{\oplus \dim T}}$. It follows from Lemma 7.3.1 and the main result of [EG23, Chapter 7]. □

7.3.3. Proposition For each tame torus T, $\mathcal{X}_{^LT}$ is equivalent to the moduli stack of continuous representations of the Weil group W_K valued in ^LT.

Proof. By the proof of Lemma 7.3.1, the objects of $\mathcal{X}_{^LT}$ are objects of $\mathcal{X}_{^LT_E} = \mathcal{X}_{\mathrm{G}_m^{\oplus \dim T}}$ equipped with tame descent data. By the main result of [EG23, Chapter 7], the stack $\mathcal{X}_{\mathrm{G}_m^{\oplus \dim T}}$ is equivalent to the stack of Weil representations for $\mathrm{G}_m^{\oplus \dim T}$; and thus by descent, $\mathcal{X}_{^LT}$ is equivalent to the stack of Weil representations for T. Alternatively, we can invoke Corollary 2.7.4, and the canonical morphism from the moduli of Weil representations to $\mathcal{X}_{^LT}$ can be constructed in the same manner it is done in [EG23, Chapter 7] for G_m. □

7.3.4. Almost étale descent

7.3.5. Étale (φ, G_K)-modules Let A be a finite type $\mathbb{Z}/p^a$-algebra. Write $W(\mathbb{C}^\flat)_A$ for the v-adic completion of $W(\mathbb{C}^\flat) \otimes_{\mathbb{Z}} A$ where v is an element of the maximal ideal of $W_a(\mathcal{O}_{\mathbb{C}}^\flat)$ whose image in $\mathcal{O}_{\mathbb{C}}^\flat$ is non-zero. (cf. [EG23, Section 2.2].)

An *étale (φ, G_K)-module with A-coefficients* is a finitely generated $W(\mathbb{C}^\flat)_A$-module M equipped with an isomorphism $\phi_M : \varphi^*M \cong M$ of $W(\mathbb{C}^\flat)_A$-modules and a $W(\mathbb{C}^\flat)_A$-semilinear action of Gal_K which is continuous and commutes with ϕ_M.

An *étale (ϕ, G_K)-module with A-coefficients and ^LG-structure* is a faithful, exact, symmetric monoidal functor F from $^f\,\mathrm{Rep}_{^LG}$ to the category of projective étale (φ, G_K)-module with A-coefficients.

7.3.6. Lemma For each finite type $\mathbb{Z}/p^a$-algebra A, the groupoid $\mathcal{Z}^{\Gamma}_{^LG}(A)$ is equivalent to the groupoid of étale (ϕ, G_K)-module with A-coefficients and ^LG-structure.

Proof. It follows from [EG23, Proposition 2.7.8] and Tannakian formalism. □

As a consequence, the objects of $\mathcal{X}_{\mathcal{Z},{}^LG}$ can be interpreted as étale (φ, G_K)-module with ^LG-structure and $\widehat{G}$-level structure.

7.3.7. Lemma We have

$$\mathcal{X}_{^LG} = \mathcal{X}_{\mathcal{Z},{}^LG}.$$

The claim holds even if K is wildly ramified over $\mathbb{Q}_p$.

Proof. The proof is identical to that of Proposition 3.7.4. The input is Corollary 2.7.4 and the Ind-finite-type-ness of both Ind-algebraic stacks. □

7.3.8. Definition Let L/K be a finite extension. Define $\mathcal{X}_{L,{}^LG}$ to be the limit-preserving stack over $\mathbb{Z}/p^a$ such that for each finitely presented $\mathbb{Z}/p^a$-algebra A, $\mathcal{X}_{L,{}^LG}(A)$ is the groupoid of étale (φ, G_L)-module with ${}^LG$-structure and $\widehat{G}$-level structure.

There stacks can be defined even if G is not assumed to be reductive. In general, let $\widehat{H}$ be a connected affine smooth group over $\operatorname{Spec}\mathbb{Z}_p$, equipped with an action of $\mathrm{Gal}(E/K)$. Write ${}^LH$ for $\widehat{H} \rtimes \mathrm{Gal}(E/K)$ and it makes sense to define $\mathcal{X}_{L,{}^LH}$.

We also define stacks $\mathcal{R}^{\gamma}_{L,{}^LG}$, $\mathcal{R}^{\Gamma}_{L,{}^LG}$, $\mathcal{R}^{\gamma}_{L,{}^LG,\widehat{G}}$ and $\mathcal{R}^{\Gamma}_{L,{}^LG,\widehat{G}}$ in a similar way (replacing $\mathbb{A}_K$ by $\mathbb{A}_L$ in the original definition).

7.3.9. Lemma (1) The first and the second diagonal of $\mathcal{X}_{L,{}^LG}$ are representable by algebraic spaces, affine and of finite presentation.

(2) The forgetful morphism

$$\mathcal{X}_{K,{}^LG} \to \mathcal{X}_{L,{}^LG}$$

is representable by algebraic spaces, affine and of finite presentation.

Proof. (1) It is true if ${}^LG = \mathrm{GL}_N$ by [EG23, Proposition 3.2.17]. By choosing an embedding ${}^LG \to \mathrm{GL}_N$ (which is a closed immersion as a scheme morphism), the diagonal of $[\mathcal{X}_{L,{}^LG}/\,\mathrm{Gal}(E/K)]$ can be embeded into the diagonal of $\mathcal{X}_{L,\mathrm{GL}_N}$ as a closed immersion (see the proof of Theorem 3.3.10).

(2) By inspecting the proof of [EG23, Lemma 3.7.5], it only makes use of the representability of the diagonal of $\mathcal{X}_{L,d}$. The same proof works verbatim for general groups. □

7.3.10. Remark Note that $\mathcal{X}_{K,{}^LG} = \mathcal{X}_{{}^LG}$ by Lemma 7.3.7. If L is tame and Galois over $\mathbb{Q}_p$, then by the independence of the splitting field (Proposition 7.2.5),

$$\mathcal{X}_{L,{}^LG} \cong \mathcal{X}_{L,{}^LG_L} = \mathcal{X}_{{}^LG_L}.$$

In particular, if $L = E$, then

$$\mathcal{X}_{E,{}^LG} \cong \mathcal{X}_{E,{}^LG_E} \cong \mathcal{X}_{E,\widehat{G}_E} = \mathcal{X}_{\widehat{G}_E}.$$

8. Herr complexes, obstruction theory and cohomology types

Let A be a finitely presented $\mathbb{Z}/p^a$-algebra and let (M, ϕ_M, γ_M) be a projective étale (φ, Γ)-module with A-coefficients. The Herr complex ([EG23, Section 5.1]) for M is defined to be the following perfect complex

$$\mathcal{C}^\bullet(M) := [0 \to M \xrightarrow{(\phi_M-1,\gamma_M-1)} M \oplus M \xrightarrow{(\gamma_M-1,1-\phi_M)} M \to 0]. \tag{1}$$

By [EG23, Theorem 5.1.29], (in the derived category of cochain complexes) the Herr complex is an algebraic interpolation of Galois cohomology.

Fix an embedding $i : \widehat{G} \hookrightarrow \mathrm{GL}(V)$. Note that i induces an embedding of tangent spaces

$$\mathrm{Lie}(i) : \mathrm{Lie}(\widehat{G}) \hookrightarrow \mathrm{End}(V).$$

Write Ad for the adjoint representation $\widehat{G} \to \mathrm{GL}(\mathrm{Lie}\,\widehat{G})$ or $\mathrm{GL}(V) \to \mathrm{GL}(\mathrm{End}(V))$.

8.1. Obstruction theory for $\mathcal{X}_{\widehat{G}_E}$

Let F be an étale (γ,Γ)-module with A-coefficients and $\widehat{G}$-structure. Let $\mathcal{E}$ be a finitely generated A-module, and write $A[\mathcal{E}] := A \oplus \mathcal{E}$ for the A-algebra with multiplication given by

$$(a,m)(a',m') := (aa', am' + a'm).$$

Denote by $\mathrm{Lift}(F, A[\mathcal{E}])$ the set of isomorphism classes of projective étale (φ,Γ)-modules F' with $A[\mathcal{E}]$-coefficients and $\widehat{G}$-structure together such that $F' \otimes_{A[\mathcal{E}]} A \cong F$.

For ease of notation write ϕ_V for $\phi_{F\times^{\widehat{G}}V}$ and write γ_V for $\gamma_{F\times^{\widehat{G}}V}$.

8.1.1. Lemma The map

$$(\dagger) \quad \begin{aligned} Z^1(C^\bullet(F \times^{\widehat{G}} \mathrm{End}(V) \otimes_A \mathcal{E})) &\to \mathrm{Lift}(F \times^{\widehat{G}} V, A[\mathcal{E}]) \\ (X,Y) &\mapsto (V \otimes_A A[\mathcal{E}], (X+1)\phi_V, (Y+1)\gamma_V)) \end{aligned}$$

induces a bijection

$$H^1(C^\bullet(F \times^{\widehat{G}} \mathrm{End}(V) \otimes_A \mathcal{E})) \xrightarrow{\cong} \mathrm{Lift}(F \times^{\widehat{G}} V, A[\mathcal{E}])$$

which respects the natural A-module structures.

Proof. It is [EG23, Lemma 5.1.35]. □

Since $F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \hookrightarrow F \times^{\widehat{G}} \mathrm{End}(V)$, there exists a cochain map $\mathcal{C}^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G}) \hookrightarrow \mathcal{C}^\bullet(F \times^{\widehat{G}} \mathrm{End}(V))$, which induces homomorphisms of cohomology groups.

8.1.2. Lemma (1) The composition

$$Z^1(C^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A \mathcal{E})) \to Z^1(C^\bullet(F \times^{\widehat{G}} \mathrm{End}(V) \otimes_A \mathcal{E})) \to \mathrm{Lift}(F \times^{\widehat{G}} V, A[\mathcal{E}])$$

factors through $\mathrm{Lift}(F, A[\mathcal{E}])$.

(2) The map $Z^1(C^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A \mathcal{E})) \to \mathrm{Lift}(F, A[\mathcal{E}])$ factors through $H^1(C^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A \mathcal{E}))$.

Proof. Choose a smooth cover $t : \mathrm{Spec}\, S \to \mathrm{Spec}\, \mathbb{A}_{E,A}$ that trivializes the $\widehat{G}$-torsor F. The transition map u_{tt} of $F \times^{\widehat{G}} V$ with respect to the cover t is an element of $\widehat{G}(S \otimes_{\mathbb{A}_{E,A}} S)$, and $\phi_V \otimes 1$, $\gamma_V \otimes 1 \in \widehat{G}(S)$.

By Lemma 2.7.7, an infinitesimal lift $F' \in \mathrm{Lift}(F, A[\mathcal{E}])$ of F necessarily becomes a trivial $\widehat{G}$-torsor over $S \otimes_A A[\mathcal{E}]$.

In this lemma, we will only consider lifts $V' \in \mathrm{Lift}(F \times^{\widehat{G}} V, A[\mathcal{E}])$ that becomes a trivial vector bundle after base change to $S \otimes_A A[\mathcal{E}]$. Moreover, we insist that the transition map of V' with respect to $t \otimes_A A[\mathcal{E}]$ is $u_{tt} \otimes 1$. By descent with respect to the cover $t \otimes_A A[\mathcal{E}]$, such a lift V' comes from a lift of F if and only if $\phi_{V'} \otimes 1$, $\gamma_{V'} \otimes 1 \in \widehat{G}(S \otimes_A A[\mathcal{E}])$.

Note that

$$\begin{aligned} (X+1)\phi_V \in \widehat{G}(S \otimes_A A[\mathcal{E}]) &\Leftrightarrow X \in F \times^{\widehat{G}} \mathrm{Lie}(\widehat{G}) \otimes_{\mathbb{A}_{E,A}} S \otimes_A \mathcal{E} \\ (Y+1)\gamma_V \in \widehat{G}(S \otimes_A A[\mathcal{E}]) &\Leftrightarrow Y \in F \times^{\widehat{G}} \mathrm{Lie}(\widehat{G}) \otimes_{\mathbb{A}_{E,A}} S \otimes_A \mathcal{E}. \end{aligned}$$ □

The lemma now follows from the explicit description of $(\dagger)$ in Lemma 8.1.1.

8.1.3. Corollary There is a canonical isomorphism of A-modules

$$H^1(C^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A \mathcal{E})) \to \mathrm{Lift}(F, A[\mathcal{E}]).$$

Proof. The homomorphism $H^1(C^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A \mathcal{E})) \to \mathrm{Lift}(F, A[\mathcal{E}])$ is constructed by Lemma 8.1.2. By [Stacks, Tag 00HN], it suffices to show it becomes an isomorphism after localized at $\bar{\mathbb{F}}_p$-points of $\mathrm{Spec}\,A$. By [Stacks, Tag 00MC] and [EG23, Theorem 5.1.29], the corollary is reduced to a Galois deformation statement which is standard. □

Next consider an arbitrary square zero thickening $0 \to I \to A' \to A \to 0$ of A, and define $\mathrm{Lift}(F, A')$ to be the set of isomorphism classes of projective étale (φ, Γ)-modules F' with A'-coefficients and $\widehat{G}$-structure together with an isomorphism $F' \otimes_{A'} A \cong F$. In the paragraph before [EG23, Lemma 5.1.34], an element

$$o_F(A') := \widetilde{\phi_V}\widetilde{\gamma_V}\widetilde{\phi_V}^{-1}\widetilde{\gamma_V}^{-1} - 1 \in \mathcal{C}^2(F \times^{\widehat{G}} \mathrm{End}(V) \otimes_A I) \tag{2}$$

is defined and [EG23, Lemma 5.1.34] states that $\mathrm{Lift}(F \times^{\widehat{G}} V, A') \neq 0$ if and only if the image of $o_F(A')$ is $H^2(\mathcal{C}^\bullet(F \times^{\widehat{G}} \mathrm{End}(V) \otimes_A I))$ is zero.

8.1.4. Lemma (1) The element $o_F(A')$ lies in $\mathcal{C}^2(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A I)$.

(2) $\mathrm{Lift}(F, A') \neq 0$ if and only if the image of $o_F(A')$ is $H^2(\mathcal{C}^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A I))$ is zero.

Proof. The proof is similar to that of Lemma 8.1.2. We choose a smooth cover $\mathrm{Spec}\,S \to \mathrm{Spec}\,\mathbb{A}_{E,A}$ and it is immediate from equation (2) that $o_F(A') \otimes_{\mathbb{A}_{E,A}} S \in \mathcal{C}^2(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G} \otimes_A I) \otimes_{\mathbb{A}_{E,A}} S$. Part (1) follows from descent and part (2) follows from the same argument used in [EG23, Lemma 5.1.34]. □

8.1.5. Proposition The stack $\mathcal{X}_{\widehat{G}_E}$ admits a nice obstruction theory in the sense of [EG23, Definition A.34].

Proof. This proposition is the natural extension of [EG23, Proposition 5.2.36]. The non-formal ingredients of the proof are furnished by Corollary 8.1.3 and Lemma 8.1.4. We still need to show that $\mathcal{C}^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G})$ is a perfect complex over A and its formation is compatible with base change. Consider the composition $\mathrm{Spec}\,A \to \mathcal{X}_{\widehat{G}_E} \to \mathcal{X}_{\mathrm{GL}(\mathrm{Lie}\,\widehat{G})}$ and note that $\mathcal{C}^\bullet(F \times^{\widehat{G}} \mathrm{Lie}\,\widehat{G})$ is the pullback of the Herr complex for the universal family of étale (φ, Γ)-modules over $\mathcal{X}_{\mathrm{GL}(\mathrm{Lie}\,\widehat{G})}$. The formal ingrediants of the proof follow from [EG23, Theorem 5.1.22]. □

8.2. Restriction of groups

Recall that $\mathbb{A}_{K,A} \hookrightarrow \mathbb{A}_{E,A}$ is a Galois cover whose Galois group is canonically identified with $\mathrm{Gal}(E/K)$. Let (M, ϕ_M, γ_M) be a projective étale (φ, Γ)-module of rank n. There is an embedding of cochain complexes

$$\mathcal{C}^\bullet(M) \hookrightarrow \mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A})$$

and the induced homomorphism of cohomology groups

$$H^i(\mathcal{C}^\bullet(M)) \to H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))$$

factors through the $\mathrm{Gal}(E/K)$-invariant subgroup $H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))^{\mathrm{Gal}(E/K)}$.

8.2.1. Lemma Let $C^\bullet$ be a perfect complex over a finite type affine scheme $X = \operatorname{Spec} A$. Fix integers $h^\bullet = (h^i)_{i\in\mathbb{Z}}$. The set of points $x : \operatorname{Spec} k \to \operatorname{Spec} A$ such that $\dim_k H^i(C^\bullet \otimes^L_A k) = h^i$ is a locally closed subset $X_{h^\bullet}$ of $|X|$.

Equip $X_{h^\bullet}$ with the reduced induced subscheme structure ([Stacks, Tag 0F2L]). Then the cohomology groups $H^i(C^\bullet|_{X_{h^\bullet}})$ is a coherent sheaf over $X_{h^\bullet}$ that is projective of rank h^i.

Proof. The first paragraph is [Stacks, Tag 0BDI]. In the proof of loc. cit. it is explained that $X_{h^\bullet}$ is locally closed when X is affine; note that if X is not affine, then $X_{h^\bullet}$ is only constructible.

For the second paragraph, we only need to establish the quasi-coherence. Let d be the largest index number such that $h_d \neq 0$. The highest degree cohomology group $H^d(C^\bullet|_{X_{h^\bullet}})$ is unconditionally quasi-coherent. By [Stacks, Tag 00NX], $H^d(C^\bullet|_{X_{h^\bullet}})$ is indeed a projective module of rank h^d. The quasi-coherence and projectivity of $H^i(C^\bullet|_{X_{h^\bullet}})$ follows from the universal coefficient theorem (which states that the failure of quasi-coherence for H^i is measure by $\operatorname{Tor}^1(H^{i+1}, -)$) and induction on i. $\square$

8.2.2. Definition A projective étale (φ,Γ)-module (M, ϕ_M, γ_M) over $\mathbb{A}_{K,A}$ is said to be *of cohomology type* $(h^i_K, h^i_E)_{i=0,1,2}$ if

$$\dim_{\bar{\mathbb{F}}_p} H^i(C^\bullet(M) \otimes^L_A \bar{\mathbb{F}}_p) = h^i_K$$

and

$$\dim_{\bar{\mathbb{F}}_p} H^i(C^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}) \otimes^L_A \bar{\mathbb{F}}_p) = h^i_E$$

for all $\bar{\mathbb{F}}_p$-points of $\operatorname{Spec} A$.

We remark that by Lemma 8.2.1, for an arbitrary projective étale (φ,Γ)-module (M, ϕ_M, γ_M) with A-coefficients, there exists a stratification $\operatorname{Spec} A_{h^\bullet}$ of $\operatorname{Spec} A_{\mathrm{red}}$ such that over each stratum, M is of a certain cohomology type.

8.2.3. Proposition Let A be a reduced finite type $\mathbb{F}$-algebra. Let (M, ϕ_M, γ_M) be a projective étale (φ,Γ)-module with A-coefficients of cohomology type $(h^i_K, h^i_E)_{i=0,1,2}$. Then the canonical map

$$H^i(\mathcal{C}^\bullet(M)) \to H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))^{\mathrm{Gal}(E/K)}$$

is an isomorphism of projective A-modules of rank h^i_K.

Proof. By the universal coefficient theorem, there is an injective homomorphism

$$H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))^{\mathrm{Gal}(E/K)} \otimes_A \bar{\mathbb{F}}_p \hookrightarrow H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}) \otimes^L_A \bar{\mathbb{F}}_p)^{\mathrm{Gal}(E/K)}$$

for any $\bar{\mathbb{F}}_p$-point of $\operatorname{Spec} A$. By Lemma 8.2.1, $H^i(\mathcal{C}^\bullet(M)) \otimes_A \bar{\mathbb{F}}_p = H^i(\mathcal{C}^\bullet(M) \otimes^L_A \bar{\mathbb{F}}_p)$. By [Ko02, Theorem 3.15] and [EG23, Theorem 5.1.29], $H^i(\mathcal{C}^\bullet(M) \otimes^L_A \bar{\mathbb{F}}_p) \cong H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}) \otimes^L_A \bar{\mathbb{F}}_p)^{\mathrm{Gal}(E/K)}$. Thus

$$H^i(\mathcal{C}^\bullet(M)) \otimes_A \bar{\mathbb{F}}_p \to H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))^{\mathrm{Gal}(E/K)} \otimes_A \bar{\mathbb{F}}_p$$

is an isomorphism for all $\bar{\mathbb{F}}_p$-points of $\operatorname{Spec} A$. By [Stacks, Tag 00NX], both $H^i(\mathcal{C}^\bullet(M))$ and $H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))^{\mathrm{Gal}(E/K)}$ are projective modules of rank h^i_K; by [Stacks, Tag 00HN], $H^i(\mathcal{C}^\bullet(M)) \to H^i(\mathcal{C}^\bullet(M \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{E,A}))^{\mathrm{Gal}(E/K)}$ is surjective and therefore must be an isomorphism (since a surjective homomorphism onto a projective module admits a section). $\square$

9. Frameable (φ, Γ)-modules with ${}^LG$-structure

Thanks to the geometric Shapiro lemma (Proposition 7.2.4), it is harmless to assume $K = \mathbb{Q}_p$. We assume E is tame over $\mathbb{Q}_p$.

Let A be a finitely presented $W(\bar{\mathbb{F}}_p)/p^a$-algebra.

9.1. Frameable ${}^LG$-torsors

There are two $\mathrm{Gal}(E/K)$-actions on $\widehat{G}_{\mathbb{A}_E} := \widehat{G} \times \operatorname{Spec} \mathbb{A}_E$.

- The first action comes from the semi-direct product ${}^LG = \widehat{G} \rtimes \mathrm{Gal}(E/K)$, and we denote it by $h \mapsto \sigma h \sigma^{-1}$ ($\sigma \in \mathrm{Gal}(E/K)$).
- The second action is the base change to $\widehat{G}_{\mathbb{A}_E}$ of the field automorphism $\sigma : E \to E$, and we denote it by $h \mapsto \sigma(h)$.

These two actions commute with each other, and we write $\sigma \cdot h$ for the composition of these two.

Consider the following morphism

$$\begin{aligned} (\widehat{G} \rtimes \mathrm{Gal}(E/K)) \times \widehat{G}_{\mathbb{A}_E} &\xrightarrow{m} \widehat{G}_{\mathbb{A}_E} \\ ((g, \sigma), h) &\mapsto g(\sigma \cdot h) \end{aligned}$$

9.1.1. Lemma The morphism m above is a group action. Therefore, $\widehat{G}_{\mathbb{A}_E}$ is an ${}^LG$-torsor over $\operatorname{Spec} \mathbb{A}_K$.

Proof. We have

$$\begin{aligned} m((g_2, \sigma_2), m((g_1, \sigma_1), h)) &= m((g_2, \sigma_2), g_1 \sigma \sigma(h) \sigma^{-1}) \\ &= g_2 \sigma_2 \sigma_2 (g_1 \sigma_1 \sigma_1(h) \sigma_1^{-1}) \sigma_2^{-1} \\ &= g_2 \sigma_2 g_1 \sigma_2^{-1} (\sigma_2 \sigma_1 \cdot h). \end{aligned}$$

□

Conversely, let T be an ${}^LG$-torsor over $\operatorname{Spec} \mathbb{A}_K$. Write S for

$$T \times^{{}^LG} \mathrm{Gal}(E/K).$$

The following diagram

$$\begin{tikzcd} \widehat{G} \times T \arrow[r, "\cong"] \arrow[d, hook] & T \times_S T \arrow[d, hook] \\ {}^LG \times T \arrow[r, "\cong"] & T \times_{\operatorname{Spec} \mathbb{A}_K} T \end{tikzcd}$$

is commutative, and T is an $\widehat{G}$-torsor over S. If $S \cong \operatorname{Spec} \mathbb{A}_E$, then T is a form of $\widehat{G}_{\mathbb{A}_E}$.

The following proposition explains the role of ${}^LG$-torsors that are isomorphic to $\widehat{G}_{\mathbb{A}_E}$.

9.1.2. Proposition Let $\operatorname{Spec} A$ be a finite type $\mathbb{Z}/p^a$-algebra, and let $(F, \phi_F, \gamma_F) \in \mathcal{X}_{K,{}^LG}(A)$. There exists a scheme-theoretically surjective morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ where B is a finite type $\mathbb{Z}/p^a$-algebra such that there exists an ${}^LG$-torsor isomorphism $F \otimes_{\mathbb{A}_{K,A}} \mathbb{A}_{K,B} \cong \widehat{G}_{\mathbb{A}_{E,B}}$.

Proof. Consider the morphism

$$F \to F \times^{{}^LG} {}^L\{*\} = F \times^{{}^LG} \mathrm{Gal}(E/K).$$

By the computation in Subsection 5.1, we have $F \times^{{}^LG} \mathrm{Gal}(E/K) \cong \operatorname{Spec} \mathbb{A}_{E,A}$. In particular, F is a $\widehat{G}$-torsor over $\operatorname{Spec} \mathbb{A}_{E,A}$. So it remains to show that F is admissible when regarded as a $\widehat{G}$-torsor over $\operatorname{Spec} \mathbb{A}_{E,A}$. The rest of the proof follows from the proof of Corollary 3.7.5. □

9.1.3. Definition Let A be a finite type $\mathbb{Z}/p^a$-algebra. An ${}^LG$-torsor T over $\mathbb{A}_{E,A}$ is said to be *frameable* if $T \cong {}^LG_{\mathbb{A}_{E,K}}$.

An ${}^LG$-torsor T over $\mathbb{A}_{K,A}$ is said to be *frameable* if $T \cong \widehat{G}_{\mathbb{A}_{E,K}}$.

Define $F^{\gamma}_{E,{}^LG}(A)$ to be the full subcategory of $\mathcal{R}^{\gamma}_{E,{}^LG}(A)$ consisting of frameable objects. Define $F^{\gamma}_{K,{}^LG}(A)$ to be the full subcategory of $\mathcal{R}^{\gamma}_{K,{}^LG}(A)$ consisting of frameable objects.

For technical reasons, we also define ${}^wF^{\gamma}_{K,{}^LG}(A)$ to be the groupoid of tuples (F, ϕ_F, γ_F) where F is a frameable ${}^LG$-torsor over $\operatorname{Spec} \mathbb{A}_{K,A}$ and ϕ_F and γ_F are defined in the usual way except that we drop the commutativity relation between ϕ_F and γ_F.

9.1.4. Lemma Assume ${}^LH \to {}^LH'$ is a group homomorphism which admits a scheme-theoretic section. Then the morphism ${}^wF^{\gamma}_{K,{}^LH}(A) \to {}^wF^{\gamma}_{K,{}^LH'}(A)$ is essentially surjective.

Proof. Note that ϕ_F and γ_F are simply elements of ${}^LH(\mathbb{A}_{K,A})$ (up to twisted conjugacy). □

9.1.5. Remark Let ${}^LP$ be a parabolic subgroup of ${}^LG$. We don't exclude the possibility ${}^LP = {}^LG$. Let $\rho : \mathrm{Gal}_K \to {}^LP(\bar{\mathbb{F}}_p)$ be an L-parameter. By Tannakian formalism, ρ corresponds to an étale (φ, Γ)-module (F, ϕ_F, γ_F) with ${}^LP$-structure over $\mathbb{A}_{K,\bar{\mathbb{F}}_p}$. By Lemma 2.7.6 and the proof of Proposition 9.1.2, F is necessarily a frameable ${}^LP$-torsor, that is, $F \cong \widehat{P}_{\mathbb{A}_{E,\bar{\mathbb{F}}_p}}$.

9.2. Frameable (φ, Γ)-modules with ${}^LG$-structure

For ease of notation, write $\mathbb{A}$ for $\mathbb{A}_{E,A}$ throughout the section. Also write $- \otimes -$ for $- \otimes_{\mathbb{Z}} -$. Fix an embedding $i : {}^LG \hookrightarrow \mathrm{GL}(V)$.

Choose a trivialization $b_0 : V \xrightarrow{\cong} \mathrm{G}_a^{\oplus n}$. Write $V_{(-)}$ for $V \otimes (-)$.

Let T^{b_0} be the trivial ${}^LG$-torsor over $\mathbb{A}$ defined by the following functor

$$R \mapsto \{\text{trivializations } b : V_R \xrightarrow{\cong} R^{\oplus n} | b = g \circ b_0 \text{ for some } g \in {}^LG(R)\}.$$

Write $\varphi^* V_{\mathbb{A}}$ for $\mathbb{A} \otimes_{\varphi,\mathbb{A}} V_{\mathbb{A}}$ (where $a \otimes bx = a\phi(b) \otimes x$). Write $\varphi^* b_0 : \varphi^* V_{\mathbb{A}} \to \mathbb{A}^{\oplus n}$ for the trivialization which sends $1 \otimes b_0^{-1}(e_i)$ to e_i (where $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{A}^{\oplus n}$).

Write $T^{\varphi^* b_0}$ for the ${}^LG$-torsor defined by

$$R \mapsto \{\text{trivializations } b : \varphi^* V_{\mathbb{A}} \xrightarrow{\cong} R^{\oplus n} | b = g \circ \varphi^* b_0 \text{ for some } g \in {}^LG(R)\}.$$

Let $f : V_{\mathbb{A}} \to V_{\mathbb{A}}$ be a φ-semi-linear map, (that is, $f(ax) = \varphi(a)f(x)$). Assume $b_0(\varphi^* f) \in T^{\varphi^* b_0}$. Then the association $b \mapsto b(\varphi^* f)$ defines a ${}^LG$-torsor morphism $T^{b_0} \to T^{\varphi^* b_0}$ which we also denote by f. Consider the following commutative diagram

$$\begin{tikzcd}
{}^LG_{\mathbb{A}} \arrow[r, "g \mapsto g \circ b_0"] \arrow[d, "f_{b_0}"] & T^{b_0} \arrow[d, "f"] \\
{}^LG_{\mathbb{A}} \arrow[r, "g \mapsto g \circ \varphi^* b_0"] & T^{\varphi^* b_0}
\end{tikzcd}$$

An straightforward calculation shows

$$f_{b_0}(I_n) = f_{b_0}((e_1|e_2|\dots|e_n)) = (b_0 f b_0^{-1} e_1|\dots|b_0 f b_0^{-1} e_n) = b_0 f b_0^{-1} I_n.$$

Notation Write $[f]_{b_0}$ for $f_{b_0}(I_n)$, and call it the coordinate matrix for the semi-linear map f under the basis b_0.

For a γ-semi-linear map $g : V \to V$, we can similarly define the ${}^L G$-torsor $T^{\gamma^* b_0}$ and the coordinate matrix $[g]_{b_0}$. Recall that the ϕ- and γ-action on $\mathbb{A}$ commute with each other. The composites $f \circ g$ and $g \circ f : V \to V$ are $\phi \circ \gamma$-semi-linear maps.

9.2.1. Lemma The coordinte matrices $[f \circ g]_{b_0}$ and $[g \circ f]_{b_0}$ are given by

$$\begin{aligned}[f \circ g]_{b_0} &= [f]_{b_0} \varphi([g]_{b_0}) \\ [g \circ f]_{b_0} &= [g]_{b_0} \gamma([f]_{b_0})\end{aligned}$$

Proof. Unwrap the definitions. □

9.2.2. Lemma (Change of basis) For $h \in {}^L G(\mathbb{A})$, we have

$$[f]_{h \circ b_0} = h[f]_{b_0} \varphi(h)^{-1}.$$

9.2.3. Lemma The groupoid $F^{\gamma}_{E, {}^L G}(A)$ is equivalent to the groupoid consisting of pairs $([\phi], [\gamma])$, where $[\phi], [\gamma] \in {}^L G(\mathbb{A})$ such that $[\phi]\varphi([\gamma]) = [\gamma]\gamma([\phi])$; and two pairs $([\phi], [\gamma])$ and $([\phi]', [\gamma]')$ are equivalent if there exists $h \in {}^L G(\mathbb{A}_A)$ such that $[\phi] = h[\phi]'\varphi(h)^{-1}$ and $[\gamma] = h[\gamma]'\gamma(h)^{-1}$.

Proof. Clear from the discussions in this subsection. □

9.2.4. Definition For ease of exposition, we will identify $F^{\gamma}_{E, {}^L G}(A)$ with the category of pairs of matrices $([\phi], [\gamma])$ in the rest of this paper.

While we call the equivalence class of $([\phi], [\gamma])$ a frameable étale (φ, γ)-module with ${}^L G$-structure, we call a representative $([\phi], [\gamma])$ a *framed étale* (φ, γ)*-module with* ${}^L G$*-structure*. A *framed étale* (φ, γ)*-module of rank* n is by definition a framed étale (φ, γ)-module with GL_n-structure.

9.2.5. Galois descent datum Recall that $\mathbb{A}_{K,A} \hookrightarrow \mathbb{A}_{E,A}$ is a Galois cover whose Galois group is canonically identified with $\mathrm{Gal}(E/K)$. To define a frameable étale (φ, γ)-module with ${}^L G$-structure over $\mathbb{A}_{K,A}$, it suffices to specify a framed étale (φ, γ)-module with ${}^L G$-structure over $\mathbb{A}_{E,A}$ which we denote by $([\phi], [\gamma])$, together with a Galois descent datum $(\varphi_\sigma)_{\sigma \in \mathrm{Gal}(E/K)}$ in the sense of [Stacks, 0CDR]. If we unravel the definitions, it means $\varphi_\sigma \in {}^L G(\mathbb{A}_{E,A})$ are matrices that satisfy

$$\begin{aligned}\varphi_\sigma[\phi]\varphi(\varphi_\sigma)^{-1} &= [\sigma^{-1}]\sigma^{-1}([\phi])\sigma^{-1}\varphi([\sigma]) \\ \varphi_\sigma[\gamma]\gamma(\varphi_\sigma)^{-1} &= [\sigma^{-1}]\sigma^{-1}([\gamma])\sigma^{-1}\gamma([\sigma])\end{aligned} \tag{3}$$

and that φ_σ satisfies the obvious cocycle condition. By the discussion in Subsection 9.1, the underlying ${}^L G$-torsor over $\operatorname{Spec} \mathbb{A}_{K,A}$ is $\widehat{G}_{\mathbb{A}_{E,A}}$. The identification $\widehat{G}_{\mathbb{A}_E} \otimes_{\mathbb{A}_K} \mathbb{A}_E \cong {}^L G_{\mathbb{A}_E}$ defines the *canonical Galois descent datum* $(\varphi_{\mathrm{can},\sigma})_{\sigma \in \mathrm{Gal}(E/K)}$.

9.2.6. Lemma The groupoid $F^{\gamma}_{K,{}^LG}(A)$ is equivalent to the groupoid consisting of pairs $([\phi],[\gamma]) \in F^{\gamma}_{E,{}^LG}(A)$, where $[\phi],[\gamma] \in {}^LG(\mathbb{A})$ are matrices that are compatible with the canonical Galois descent datum $(\varphi_{\mathrm{can},\sigma})_{\sigma\in \mathrm{Gal}(E/K)}$ in the sense of equation (3); and morphisms are morphisms that are compatible with the canonical Galois descent datum.

Proof. Clear from the discussions in this subsection. □

The objects of $F^{\gamma}_{K,{}^LG}(A)$ should admit a more explicit description. We are satisfied with Lemma 9.2.6 since it suffices for our purposes.

9.2.7. Proposition For each $(F,\phi_F,\gamma_F) \in \mathcal{X}_{K,{}^LG}(A)$, there exists a finite type $\mathbb{Z}/p^a$-algebra B and a scheme-theoretically surjective morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $(F,\phi_F,\gamma_F)\otimes_{\mathbb{A}_{K,A}}\mathbb{A}_{K,B}$ corresponds to an object of $F^{\gamma}_{K,{}^LG}(A)$.

Proof. It is a reformulation of Proposition 9.1.2. □

9.3. A framed version of Herr complexes

Let (M,ϕ_M,γ_M) be a frameable étale (φ,Γ)-module of rank n over $\mathbb{A}_{K,A}$. Let $([\phi],[\gamma])$ be a framed (φ,γ)-module over $\mathbb{A}_{E,A}$ of rank n which corresponds to $M\otimes_{\mathbb{A}_{K,A}}\mathbb{A}_{E,A}$.

The goal of this subsection is to classify all extensions of $([\phi],[\gamma])$ by the trivial (φ,γ)-module. Such an extension can be described by a pair of matrices (X,Y) of shape

$$X=\begin{bmatrix}[\phi] & * \\ 0 & 1\end{bmatrix}, Y=\begin{bmatrix}[\gamma] & * \\ 0 & 1\end{bmatrix},$$

such that $X\varphi(Y)=Y\gamma(X)$. Put

$$X_L:=\begin{bmatrix}[\phi] & 0 \\ 0 & 1\end{bmatrix}, Y_L:=\begin{bmatrix}[\gamma] & 0 \\ 0 & 1\end{bmatrix},$$

and set $X_U := X_L^{-1}X$, and $Y_U := Y_L^{-1}Y$.

Write $\begin{bmatrix}0 & x_U \\ 0 & 0\end{bmatrix} := \log(X_U) := (X_U - I)$, and $\begin{bmatrix}0 & y_U \\ 0 & 0\end{bmatrix} := \log(Y_U)$. The condition $X\varphi(Y)=Y\gamma(X)$ can be rewritten as $(\varphi([\gamma])^{-1}-\gamma)x_U + (\varphi-\gamma([\phi])^{-1})y_U = 0$. So we have proved the following.

9.3.1. Proposition The first cohomology of the following complex

$$\mathbb{A}^{\oplus n} \xrightarrow{(\varphi-[\phi]^{-1},\gamma-[\gamma]^{-1})} \mathbb{A}^{\oplus n}\oplus\mathbb{A}^{\oplus n} \xrightarrow{(\gamma-\phi([\gamma])^{-1},\gamma([\phi])^{-1}-\varphi)} \mathbb{A}^{\oplus n}$$

classifies extensions of $([\phi],[\gamma])$ by the trivial (φ,γ)-module, up to equivalences.

9.3.2. Framed Herr complexes It is clear that the cochain complex

$$C^{\bullet}_{E,\mathrm{Herr}}([\phi],[\gamma]) := [\mathbb{A}^{\oplus n} \xrightarrow{(\varphi-[\phi]^{-1},\gamma-[\gamma]^{-1})} \mathbb{A}^{\oplus n}\oplus\mathbb{A}^{\oplus n} \xrightarrow{(\gamma-\phi([\gamma])^{-1},\gamma([\phi])^{-1}-\varphi)} \mathbb{A}^{\oplus n}]$$

is isomorphic (not just quasi-isomorphic) to the Herr complex $C^{\bullet}(M\otimes_{\mathbb{A}_{K,A}}\mathbb{A}_{E,A})$ defined in Equation (1).

The cochain complex $C^{\bullet}_{E,\mathrm{Herr}}([\phi],[\gamma])$ descends to a cochain complex $C^{\bullet}_{K,\mathrm{Herr}}([\phi],[\gamma])$ (via the canonical Galois descent datum) which is isomorphic to the Herr complex $C^{\bullet}(M)$.

9.4. Higher cup products

Let $\mathfrak{P}$ be an abstract group which can be written as a semi-direct product $\mathfrak{L} \ltimes \mathfrak{U}$. Write $\mathrm{ad} : \mathfrak{L} \to \mathrm{Aut}(\mathfrak{U})$ for the conjugation action. Let $\mathfrak{Z} \subset \mathfrak{U}$ be the center of $\mathfrak{U}$. Assume there is a group automorphism $\gamma : \mathfrak{P} \to \mathfrak{P}$ such that $\gamma(\mathfrak{U}) \subset \mathfrak{U}$ and $\gamma(\mathfrak{Z}) \subset \mathfrak{Z}$; and there is a group endomorphism $\varphi : \mathfrak{P} \to \mathfrak{P}$ such that $\varphi(\mathfrak{U}) \subset \mathfrak{U}$ and $\varphi(\mathfrak{Z}) \subset \mathfrak{Z}$.

For $\alpha, \beta \in \mathfrak{P}$, define $\lambda(\alpha, \beta) := \gamma(\alpha)^{-1}\beta^{-1}\alpha\varphi(\beta)$.

9.4.1. Lemma Write $\alpha = \alpha_l\alpha_u$ where $\alpha_l \in \mathfrak{L}$ and $\alpha_u \in \mathfrak{U}$. Write $\beta = \beta_l\beta_u$ where $\beta_l \in \mathfrak{L}$ and $\beta_u \in \mathfrak{U}$. Assume $\lambda(\alpha_l, \beta_l) = 1$.

(1) We have

$$\lambda(\alpha, \beta) = \gamma(\alpha_u)^{-1}\,\mathrm{ad}_{\gamma(\alpha_l)}(\beta_u^{-1})\,\mathrm{ad}_{\varphi(\beta_l)^{-1}}(\alpha_u)\varphi(\beta_u).$$

(2) Let $z, z' \in \mathfrak{Z}$. We have

$$\lambda(\alpha z, \beta z') = \lambda(\alpha, \beta)\gamma(z)^{-1}\,\mathrm{ad}_{\gamma(\alpha_l)}(z')^{-1}\,\mathrm{ad}_{\varphi(\beta_l)^{-1}}(z)\varphi(z').$$

So $\lambda(\alpha z, \beta z')\lambda(\alpha, \beta)^{-1} \in \mathfrak{Z}$. Since $\mathfrak{Z}$ is an abelian group, we switch to the additive notation, and write

$$\lambda(\alpha z, \beta z')\lambda(\alpha, \beta)^{-1} = (-\gamma + \mathrm{ad}_{\varphi(\beta_l)^{-1}})(z) + (\varphi - \mathrm{ad}_{\gamma(\alpha_l)})(z')$$

Proof. Both (1) and (2) are proved by fully expanding the left-hand side and regroup terms. □

Now let ${}^LP$ be a parabolic subgroup of ${}^LG$. Write ${}^LP = {}^LL \ltimes U$ where ${}^LL$ is a Levi subgroup and U is the unipotent radical. We have the upper central series

$$1 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

where U_{i+1}/U_i is the center of ${}^LP/U_i$.

Write $\mathrm{ad}^i : {}^LL \to \mathrm{GL}(U_i/U_{i-1})$ for the adjoint action.

Let A be a finite type $\mathbb{Z}/p^a$-algebra. Let $([\phi_i], [\gamma_i]) \in F^{\gamma}_{K,{}^LP/U_i}(A)$. Write $[\phi_i] = [\phi_L][\phi_U^i]$ where $[\phi_L] \in {}^LL(\mathbb{A})$ and $[\phi_U^i] \in (U/U_i)(\mathbb{A})$; and write $[\gamma_P] = [\gamma_L][\gamma_U^i]$ where $[\gamma_L] \in {}^LL(\mathbb{A})$ and $[\gamma_U^i] \in (U/U_i)(\mathbb{A})$. Fix an isomorphism $b : U_i/U_{i-1} \cong \mathrm{G}_a^{\oplus s}$. By abuse of notation, write $\mathrm{ad}^i_{[\phi_L]}$ for the matrix $b \circ \mathrm{ad}^i_{[\phi_L]} \circ b^{-1} \in \mathrm{GL}(\mathrm{G}_a^{\oplus s})$.

9.4.2. Lemma We have

$$(\mathrm{ad}^i_{[\phi_L]}, \mathrm{ad}^i_{[\gamma_L]}) \in F^{\gamma}_{K,\mathrm{GL}(\mathrm{G}_a^{\oplus s})}(A).$$

Proof. We need to show $\mathrm{ad}^i_{[\phi_L]}\,\varphi(\mathrm{ad}^i_{[\gamma_L]}) = \mathrm{ad}^i_{[\gamma_L]}\,\gamma(\mathrm{ad}^i_{[\phi_L]})$, which follows from $[\phi_L]\varphi([\gamma_L]) = [\gamma_L]\gamma([\phi_L])$ once we show $\varphi(\mathrm{ad}^i_{[\gamma_L]}) = \mathrm{ad}^i_{\varphi([\gamma_L])}$.

Write A_{ij} for the i,j-th entry of the matrix $\mathrm{ad}^i_{[\gamma_L]}$. Write $e_i : \mathrm{G}_a \to \mathrm{G}_a^{\oplus s}$ for the embedding into the i-th direct summand. We have

$$[\gamma_L]b^{-1}(e_j)[\gamma_L]^{-1} = \sum_{i=1}^{s} A_{ij}b^{-1}(e_i).$$

Since b commutes with φ, we have $\varphi(\mathrm{ad}^i_{[\gamma_L]}) = \mathrm{ad}^i_{\varphi([\gamma_L])}$. □

Consider the Herr complex $C^{\bullet}_{E,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]}, \mathrm{ad}^i_{[\gamma_L]})$.

9.4.3. Lemma Let $[\phi_{i-1}]$ and $[\phi_{i-1}]'$ ($[\gamma_{i-1}]$ and $[\gamma_{i-1}]'$, respectively) be two elements of $P/U_{i-1}(\mathbb{A})$ which lifts of $[\phi_i]$ ($[\gamma_i]$, respectively). We have

$$\lambda([\phi_{i-1}],[\gamma_{i-1}])\lambda([\phi_{i-1}]',[\gamma'_{i-1}])^{-1} \in B^2_{E,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]}).$$

Proof. It follows from Lemma 9.4.1 and the definitions. □

9.4.4. Definition Let $([\phi_i],[\gamma_i]) \in F^{\gamma}_{K,{}^LP/U_i}(A)$. Let $[\phi_{i-1}]$ and $[\gamma_{i-1}]$ be two elements of $P/U_{i-1}(\mathbb{A})$ which lifts of $[\phi_i]$ and $[\gamma_i]$, respectively. The element

$$[\lambda([\phi_{i-1}],[\gamma_{i-1}])] \in H^2_{E,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$$

does not depend on the choice of lifts $[\phi_{i-1}]$ and $[\gamma_{i-1}]$. Denote $[\lambda([\phi_{i-1}],[\gamma_{i-1}])]$ by $\mu([\phi_i],[\gamma_i])$ and call it the *generalized cup product* of $([\phi_i],[\gamma_i])$.

9.4.5. Lemma Let $([\phi_i],[\gamma_i]) \in F^{\gamma}_{K,{}^LP/U_i}(A)$. Assume $(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$ defines an étale (φ,Γ)-module of a certain cohomology type in the sense of 8.2.2.

Then the generalized cup product $\mu([\phi_i],[\gamma_i])$ lies in $H^2_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$, and vanishes if and only if $([\phi_i],[\gamma_i])$ admits a lift in $F^{\gamma}_{K,{}^LP/U_{i-1}}(A)$.

Proof. It is clear that the image of

$$\lambda([\phi_{i-1}],[\gamma_{i-1}]) \in C^2_{E,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$$

in $H^2_{E,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$ vanishes if and only if $([\phi_i],[\gamma_i])$ admits a lift in $F^{\gamma}_{E,{}^LP/U_{i-1}}(A)$. By Lemma 9.1.4, $([\phi_i],[\gamma_i])$ admits a lift in ${}^wF^{\gamma}_{K,{}^LP/U_{i-1}}(A)$, and thus we have

$$\lambda([\phi_{i-1}],[\gamma_{i-1}]) \in C^2_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]}).$$

By Proposition 8.2.3,

$$H^2_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]}) \hookrightarrow H^2_{E,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$$

is injective. Therefore the image of $\lambda([\phi_{i-1}],[\gamma_{i-1}])$ in $H^2_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$ also vanishes. Since $\lambda([\phi_{i-1}],[\gamma_{i-1}]) \in B^2_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$, we can write $\lambda([\phi_{i-1}],[\gamma_{i-1}]) = d([\phi_z],[\gamma_z])$ for some $([\phi_z],[\gamma_z]) \in C^1_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]},\mathrm{ad}^i_{[\gamma_L]})$. It is easy to verify that $([\phi_{i-1}][\phi_z]^{-1},[\gamma_{i-1}][\gamma_z]^{-1}) \in F^{\gamma}_{K,{}^LP/U_{i-1}}(A)$. □

10. Noetherianness and algebraicity of $\mathcal{X}_{{}^LG}$

10.1. Parabolic étale (φ,Γ)-modules The following lemma is the motivation of Definition 10.1.2.

10.1.1. Lemma Let A be a finitely presented $\mathbb{Z}/p^a$-algebra and let $\mathrm{Spec}\,A \to \mathcal{X}_{K,{}^LG}$ be a morphism. There exists a finitely presented affine $\mathbb{Z}/p^a$-scheme $\mathrm{Spec}\,B = \mathrm{Spec}\,B_1 \coprod \mathrm{Spec}\,B_2 \coprod \cdots \coprod \mathrm{Spec}\,B_s$ and a scheme-theoretically surjective morphism $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$ such that the composite $\mathrm{Spec}\,B_i \to \mathcal{X}_{K,{}^LG}$ corresponds to an étale (φ,Γ)-module F with ${}^LG$-structure and B_i-coefficients which is frameable and of a certain cohomology type.

Proof. See the remarks in Definition 8.2.2 and Proposition 9.1.2. □

10.1.2. Definition An étale (φ, Γ)-module with ${}^LG$-structure and A coefficients is said to be *basic* if it is frameable and of a certain cohomology type. By abuse of notation, we say the corresponding morphism $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LG}$ is basic.

10.1.3. Coarse moduli space Let $\mathcal{X}$ be an algebraic stack. The *coarse moduli sheaf* of X is the sheafification of the presheaf

$$\mathcal{X}^C : \operatorname{Spec} A \mapsto \{\text{isomorphism classes of objects of } \mathcal{X}(\operatorname{Spec} A)\}.$$

We say $\mathcal{X}^C$ is the *coarse moduli space* of $\mathcal{X}$ if it is representable by an algebraic space.

10.1.4. Definition Let ${}^LP$ be a parabolic subgroup of ${}^LG$, with unipotent radical U and Levi subgroup ${}^LL$. Write $1 = U_0 \subset U_1 \subset \cdots \subset U_n = U$ for the upper central series of U. Write $\mathrm{ad}^i : {}^LL \to \mathrm{GL}(U_i/U_{i-1})$ for the adjoint action. A morphism $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_i}$ is said to be *U-basic* if the composition

$$\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_i} \to \mathcal{X}_{K,{}^LL} \xrightarrow{\mathrm{ad}^s} \mathcal{X}_{K,\mathrm{GL}(U_s/U_{s-1})}$$

is basic for $1 \le s \le n$.

10.1.5. Lemma Let A be a reduced, finite type $\mathbb{Z}/p^a$ algebra, and let $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_i}$ be a U-basic morphism. Write $\mathcal{X}_A$ for

$$\operatorname{Spec} A \underset{\mathcal{X}_{K,{}^LP/U_i}}{\times} \mathcal{X}_{K,{}^LP/U_{i-1}}.$$

The image of $\mathcal{X}_A(\bar{\mathbb{F}}_p)$ in $(\operatorname{Spec} A)(\bar{\mathbb{F}}_p)$ is a closed subset of $(\operatorname{Spec} A)(\bar{\mathbb{F}}_p)$, and thus determines a closed subset Z of $\operatorname{Spec} A$. Equip Z with the induced reduced scheme structure and write $Z = \operatorname{Spec} A/I$. Then $\operatorname{Spec} A/I \to \mathcal{X}_{K,{}^LP/U_i}$ factors through a basic morphism $\operatorname{Spec} A/I \to \mathcal{X}_{K,{}^LP/U_{i-1}}$.

Proof. Since $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_i}$ is U-basic, the cohomology of Herr complexes

$$H^2(C^\bullet(\mathrm{ad}^i))$$

is a locally free coherent sheaf over $\operatorname{Spec} A$. Write $\mathcal{Z}$ for $\underline{\operatorname{Spec}}\operatorname{Sym}(H^2(C^\bullet(\mathrm{ad}^i)))$. The morphism $\mathcal{Z} \to \operatorname{Spec} A$ is a vector bundle of rank r. The formation of the generalized cup product (Definition 9.4.4) is compatible with base change, and thus (via the functor of points interpretation of schemes) defines a morphism $\mu : \operatorname{Spec} A \to \mathcal{Z}$ by Lemma 9.4.5. We can check that μ is a closed immersion, using, for example, the Noetherian valuative criterion for properness. Write $\mathcal{Z}_0 \cong \operatorname{Spec} A$ for the zero section of the vector bundle $\mathcal{Z}$ over $\operatorname{Spec} A$. The set Z is the intersection of $\mathcal{Z}_0$ with the image of μ, and is thus a closed subset of $\mathcal{Z}_0$.

It remains to show the morphism $Z = \operatorname{Spec} A/I \to \mathcal{X}_{K,{}^LP/U_i}$ admits a lift to $\mathcal{X}_{K,{}^LP/U_{i-1}}$. We adopt the notation used in Definition 9.4.4 and work with framed étale (φ, Γ)-modules. The morphism $\operatorname{Spec} A/I \to \mathcal{X}_{K,{}^LP/U_i}$ can be presented by an object $([\phi_i], [\gamma_i]) \in F^\gamma_{K,{}^LP/U_i}(A/I)$. Let $([\phi_{i-1}], [\gamma_{i-1}]) \in {}^wF^\gamma_{K,{}^LP/U_{i-1}}(A/I)$ be an arbitrary lift of $([\phi_i], [\gamma_i])$. Since $\lambda([\phi_{i-1}], [\gamma_{i-1}]) \in B^2_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]}, \mathrm{ad}^i_{[\gamma_L]})$ by Lemma 9.4.5 and the definition of $\operatorname{Spec} A/I$, there exists an element $([\phi_z], [\gamma_z]) \in C^1_{K,\mathrm{Herr}}(\mathrm{ad}^i_{[\phi_L]}, \mathrm{ad}^i_{[\gamma_L]})$ such that $d([\phi_z], [\gamma_z]) = \lambda([\phi_{i-1}], [\gamma_{i-1}])$. The pair $([\phi_{i-1}][\phi_z]^{-1}, [\gamma_{i-1}][\gamma_z]^{-1}) \in F^\gamma_{K,{}^LP/U_{i-1}}(A)$, and thus defines a basic morphism $\operatorname{Spec} A/I \to \mathcal{X}_{K,{}^LP/U_{i-1}}$. $\square$

10.1.6. Corollary The coarse moduli sheaf of the stack $\mathcal{X}_A = \operatorname{Spec} A \underset{\mathcal{X}_{K,{}^LP/U_i}}{\times} \mathcal{X}_{K,{}^LP/U_{i-1}}$ considered in Lemma 10.1.5 is representable by a finite type scheme. Indeed, after possibly replacing $\operatorname{Spec} A$ by a Zariski cover, there exists a commutative diagram of scheme morphisms

$$\begin{tikzcd}
(\mathcal{X}_A)^C \arrow[rr, "\cong"] \arrow[dr] & & \operatorname{Spec} A/I \times \mathrm{G}_a^{\oplus m} \arrow[dl] \\
& \operatorname{Spec} A &
\end{tikzcd}$$

Proof. Since $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_i}$ is U-basic, the cohomology of Herr complexes

$$H^1(C^\bullet(\mathrm{ad}^i))$$

is a locally free coherent sheaf over $\operatorname{Spec} A$. After possibly replacing $\operatorname{Spec} A$ by a Zariski cover, we can assume $H^1(C^\bullet(\mathrm{ad}^i))$ is free of rank m. It is also harmless to replace A by A/I. Let $\{x_1, \ldots, x_m\}$ be a basis of $H^1(C^\bullet(\mathrm{ad}^i))$. As in the proof of Lemma 10.1.5, we will freely use the notation introduced in Definition 9.4.4.

The morphism $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_i}$ is presented by a pair $([\phi_i], [\gamma_i]) \in F^{\gamma}_{K,{}^LP/U_i}(A)$. Each x_k can be presented by a pair $([\phi_k^h], [\gamma_k^h]) \in Z^1(C^\bullet(\mathrm{ad}^i_{[\phi_L]}, \mathrm{ad}^i_{[\gamma_L]}))$. Lemma 10.1.5 shows that there exists a lift $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LP/U_{i-1}}$, which is presented by a pair $([\phi_{i-1}], [\gamma_{i-1}]) \in F^{\gamma}_{K,{}^LP/U_{i-1}}(A)$. We construct a morphism

$$f : \operatorname{Spec} A \times \mathrm{G}_a^{\oplus m} \to (\mathcal{X}_A)^C$$

sending $(t_1, \ldots, t_m)$ to

$$([\phi_{i-1}] \prod_{k=1}^{m} [t_k \phi_k^h], [\gamma_{i-1}] \prod_{k=1}^{m} [t_k \gamma_k^h]).$$

It is clear that f is a sheaf-theoretic bijection. $\square$

10.1.7. Corollary $\mathcal{X}_A$ is isomorphic to the quotient stack $[(\mathcal{X}_A)^C / H^0(C^\bullet(\mathrm{ad}^i))]$.

Proof. Morphisms in the category $\mathcal{X}_A$ are morphisms in $\mathcal{X}_{K,{}^LP/U_{i-1}}$ that are sent to the identity morphism in $\mathcal{X}_{K,{}^LP/U_i}$; thus they are conjugations by unipotent elements in U_{i-1}/U_i. Calculation shows the stabilizer subgroup of the group unipotent conjugations is identified with $H^0(C^\bullet(\mathrm{ad}^i))$. $\square$

10.1.8. Proposition Let $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LL}$ be a basic morphism. The fiber product

$$\operatorname{Spec} A_{\mathrm{red}} \underset{\mathcal{X}_{K,{}^LL}}{\times} \mathcal{X}_{K,{}^LP}$$

is representable by a finite type reduced algebraic stack $\mathcal{X}_{A,P}$ over $\operatorname{Spec} \mathbb{F}$. Moreover, the morphism $\mathcal{X}_{A,P} \to \mathcal{X}_{K,{}^LP}$ is relatively representable by basic morphisms.

Proof. Combine Lemma 10.1.5, Corollary 10.1.6 and Corollary 10.1.7. $\square$

10.1.9. Lemma If there exists a reduced finite type $\mathbb{Z}$-algebra A and a surjective morphism $\operatorname{Spec} B \to \mathcal{X}_{K,{}^LL}$, then there exists a reduced finite type $\mathbb{Z}$-algebra B and a surjective morphism $\operatorname{Spec} B \to \mathcal{X}_{K,{}^LP}$.

Proof. By Lemma 10.1.1, there exists a scheme-theoretically surjective, basic morphism $\operatorname{Spec} A \to \mathcal{X}_{K,{}^LL}$. By Proposition 10.1.8, $\operatorname{Spec} A_{\mathrm{red}} \underset{\mathcal{X}_{K,{}^LL}}{\times} \mathcal{X}_{K,{}^LP}$ is a finite type algebraic stack over $\mathbb{Z}/p^a$. Choose a smooth cover $\operatorname{Spec} B$ of $\operatorname{Spec} A_{\mathrm{red}} \underset{\mathcal{X}_{K,{}^LL}}{\times} \mathcal{X}_{K,{}^LP}$ and we are done. □

10.2. The representability of $\mathcal{X}_{{}^LG}$

The only non-formal part of the proof is to construct finitely many Noetherian closed substacks of $\mathcal{X}_{{}^LG}$ whose $\bar{\mathbb{F}}_p$-points jointly exhaust the $\bar{\mathbb{F}}_p$-points of $\mathcal{X}_{{}^LG}$.

10.2.1. Theorem (1) The reduced Ind-algebraic stack $\mathcal{X}_{{}^LG,\mathrm{red}}$ is an algebraic stack, of finite presentation over $\mathbb{F}_p$.

Assume either $a = 1$ or $E/\mathbb{Q}_p$ is a tame extension.

(2) $\mathcal{X}_{{}^LG}$ admits a nice obstruction theory in the sense of [EG23, Definition A.32].

(3) $\mathcal{X}_{{}^LG}$ is a Noetherian formal algebraic stack over $\mathbb{Z}/p^a$.

Proof. By Lemma 7.3.9, it suffices to show the representability of $\mathcal{X}_{E,{}^LG} \cong \mathcal{X}_{{}^LG_E}$. By the independence of the splitting field (Proposition 7.2.5), we can assume G is a split group and $\widehat{G} = {}^LG$.

(1) We apply induction on the dimension of G. The base case in our induction is when G is a torus (Lemma 7.3.2).

By induction, we assume for all proper Levi subgroups L of G, $\mathcal{X}_{{}^LL,\mathrm{red}}$ is an algebraic stack of finite presentation over $\mathbb{F}_p$. By [L23], if a continuous homomorphism $\mathrm{Gal}_K \to {}^LG(\bar{\mathbb{F}}_p)$ does not factor through any proper ${}^LP$, it must factor through ${}^LS$ where S is an elliptic tame torus of G. For each stable conjugacy class $[S]$ of elliptic tame tori, choose a scheme-theoretically dominant morphism $\operatorname{Spec} B_S \to \mathcal{X}_{{}^LS,\mathrm{red}}$. By Lemma 10.1.9, for each geometric conjugacy class of proper parabolic P of G, we can choose a surjective morphism $\operatorname{Spec} B_P \to \mathcal{X}_{K,{}^LP}$. Write

$$\operatorname{Spec} B := (\coprod_S \operatorname{Spec} B_S) \amalg (\coprod_P \operatorname{Spec} B_P),$$

which is a finite type (=finitely presented) $\mathbb{Z}/p^a$-affine scheme. Recall that $\mathcal{X}_{{}^LG} \cong \varinjlim_m \mathcal{X}_{{}^LG,m}$ is a directed colimit of algebraic stacks of finite presentation over $\mathbb{F}_p$ (see 6.1.5 for the Ind-presentation), where all transition maps are closed immersions. By [EG21, Lemma 4.2.6], the morphism $\operatorname{Spec} B \to \mathcal{X}_{{}^LG}$ factors through one of $\mathcal{X}_{{}^LG,m}$. Write $\mathcal{Z}$ for the scheme-theoretic image of $\operatorname{Spec} B$ in $(\mathcal{X}_{{}^LG,m})_{\mathrm{red}}$. By the definition of scheme-theoretic image, $\mathcal{Z} \hookrightarrow \mathcal{X}_{{}^LG,\mathrm{red}}$ is a closed immersion. Let $\operatorname{Spec} A \to \mathcal{X}_{{}^LG,\mathrm{red}}$ be a morphism where A is a reduced finitely presented $\mathbb{Z}/p^a$-algebra. By the definition of $\operatorname{Spec} B$, $\mathcal{Z} \times_{\mathcal{X}_{{}^LG,\mathrm{red}}} \operatorname{Spec} A \to \operatorname{Spec} A$ is a surjective closed immersion of reduced finitely presented $\mathbb{Z}/p^a$-affine scheme, and is thus an isomorphism. Since $\mathcal{X}_{{}^LG,\mathrm{red}}$ is limit-preserving, $\mathcal{Z} \hookrightarrow \mathcal{X}_{{}^LG,\mathrm{red}}$ is an isomorphism.

(2) It is Proposition 8.1.5.

(3) By (1) and Theorem 7.1.2, $\mathcal{X}_{{}^LG}$ is, when regarded as a locally countably indexed algebraic stack (see [EG23, Proposition A.9]), Ind-locally of finite type over $\mathbb{Z}/p^a$ ([EG23, Remark A.20]). It follows from (2) and [EG23, Theorem A.33] that $\mathcal{X}_{{}^LG}$ is locally Noetherian and hence Noetherian. □

10.2.2. Corollary If G splits over a tame extension of $\mathbb{Q}_p$, then $\mathcal{X}_{^LG}$ is a Noetherian formal algebraic stack over $\operatorname{Spf}\mathbb{Z}_p$.

Proof. It follows from Theorem 10.2.1 and Proposition [EG23, A.13]. $\square$

10.3. Some functorial properties of $\mathcal{X}_{^LG}$

We record some standard functorial properties in this subsection which will be used frequently in a subsequent paper to calculate the dimension of $\mathcal{X}_{^LG}$.

10.3.1. Definition Let $\mathcal{X}$ and $\mathcal{Y}$ be Ind-algebraic stacks that are of Ind-finite type. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is said to be *of strongly Ind-finite type* if we can write $\mathcal{X} = \varinjlim_m \mathcal{X}_m$ and $\mathcal{Y} = \varinjlim_m \mathcal{Y}_m$ where transition maps are closed immersions such that $f(\mathcal{X}_m) \subset \mathcal{Y}_m$ and f is of finite type when restricted to each $\mathcal{X}_m$.

10.3.2. Lemma Let $f : {}^LG \to {}^LH$ be an L-homomorphism. Then the canonical morphism

$$\mathcal{X}_{^LG} \to \mathcal{X}_{^LH}$$

is of strongly Ind-finite type.

Proof. See Paragragh 6.1.5. $\square$

10.3.3. Lemma Let G_1, G_2 and G_3 be connected reductive groups, together with L-homomorphisms ${}^LG_1 \to {}^LG_3$ and ${}^LG_2 \to {}^LG_3$. Let H be a reductive group such that ${}^LH \cong {}^LG_1 \underset{^LG_3}{\times} {}^LG_2$.

There is a canonical isomorphism

$$\mathcal{X}_{^LG_1} \underset{\mathcal{X}_{^LG_3}}{\times} \mathcal{X}_{^LG_2} \xrightarrow{\cong} \mathcal{X}_{^LH}.$$

Proof. By the previous lemma, there is a commutative diagram

$$\begin{tikzcd} \mathcal{X}_{^LH} \arrow[r] \arrow[d] & \mathcal{X}_{^LG_1} \arrow[d] \\ \mathcal{X}_{^LG_2} \arrow[r] & \mathcal{X}_{^LG_3} \end{tikzcd}$$

So there is a morphism $\mathcal{X}_{^LH} \to \mathcal{X}_{^LG_1} \underset{\mathcal{X}_{^LG_3}}{\times} \mathcal{X}_{^LG_2}$ by the definition of fiber product. It remains to show it is an isomorphism. The Lemma follows from Theorem 2.7.3 and Theorem 7.1.2. $\square$

A. Drinfeld's descent theory and twisted affine Grassmannians

A.1. Descending $\mathcal{G}$-torsors Let R be a ring which is not necessarily noetherian.

A.1.1. Lemma (1) Let $M_1 \to M_2 \to M_3$ be a sequence of $R((u))$-modules. Let $R \to S$ be an fpqc ring map. Assume $0 \to M_1 \otimes_{R((u))} S((u)) \to M_2 \otimes_{R((u))} S((u)) \to M_3 \otimes_{R((u))} S((u)) \to 0$ is a short exact sequence of finitely generated projective $S((u))$-modules, then $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finitely generated projective $R((u))$-modules.

(2) The same is true for $R[[u]]$.

Proof. (1) By [EG21, Theorem 5.1.18], each of M_i is finitely generated projective. By Lemma 2.5.4, it suffices to show that if $f : \operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, then each closed point of $\operatorname{Spec} R((u))$ is contained in the image of f. Let $\mathfrak{p}$ be a maximal ideal of $R((u))$. Then $\mathfrak{p}_0 := \{\text{Leading coefficient of } f | f \in \mathfrak{p}\}$ is a proper ideal of R. Since $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, $\mathfrak{p}_0 S \neq S$. Hence $\mathfrak{p}S((u)) \neq S((u))$, and $\operatorname{Spec} S((u)) \times_{\operatorname{Spec} R((u))} \operatorname{Spec} R((u))/\mathfrak{p}$ is non-empty.

(2) The proof is completely similar to that of (1). $\square$

Let $\mathcal{G}$ be a smooth affine group scheme of finite type over $\mathcal{O}$ whose connected components all admit an integral point. For an $\mathcal{O}$-scheme X, let $\mathcal{G}\operatorname{Tor}_X$ be the category of $\mathcal{G}$-torsors over X.

Let ${}^f\operatorname{Rep}_G(\mathcal{O})$ be the category of linear representations of $\mathcal{G}$ on finitely generated projective $\mathcal{O}$-modules.

We recall the following theorem of [Lev15, Theorem 2.1.1].

A.1.2. Theorem Let X be an algebraic space over $\operatorname{Spec} \mathcal{O}$. Then there is an equivalence of categories $F : \mathcal{G}\operatorname{Tor}_X \xrightarrow{\sim} [{}^f\operatorname{Rep}_G(\mathcal{O}), \operatorname{Vect}_X]^{\otimes}$, which commutes with arbitrary base change on X.

Proof. [Lev15, 2.1.1] or [Lev13, 2.5.2] proves this when $\mathcal{G}$ is a group with connected geometric fibres and when X is a scheme. However, the connectivity requirement is only used to guarantee that $\mathcal{O}_{\mathcal{G}}$ is $\mathcal{O}$-projective.

The algebraic space case follows formally from the scheme case by descent. $\square$

We fix some notations. Let P be a $\mathcal{G}$-torsor over X. Let $r : \mathcal{G} \to \operatorname{GL}(V)$ be a representation of $\mathcal{G}$. Set $P \times^{\mathcal{G}} V := F(P)(r)$. Also denote by $P \times^{\mathcal{G}} \operatorname{GL}(V)$ the principal bundle associated to $P \times^{\mathcal{G}} V$.

A.1.3. Lemma Let $R \to S$ be an fpqc ring map.

(1) The category of $\mathcal{G}$-torsors over $\operatorname{Spec} R((u))$ is equivalent to the category of $\mathcal{G}$-torsors over $\operatorname{Spec} S((u))$ with descent datum.

(2) The category of $\mathcal{G}$-torsors over $\operatorname{Spec} R[[u]]$ is equivalent to the category of $\mathcal{G}$-torsors over $\operatorname{Spec} S[[u]]$ with descent datum.

Proof. Follows formally from Lemma A.1.1 and Theorem A.1.2. $\square$

A.2. The $\mathcal{F}$-twisted affine Grassmannian This main result of this section is an elaboration of the remark under Propositon 3.8 of [Dri06]. The terminology "$\mathcal{F}$-twisted affine Grassmannian" also comes from [Dri06].

Let R be an arbitrary ring. Let $\mathcal{G}$ be a smooth affine $\mathcal{O}$-group scheme of finite type.

We fix some notations. Write D_R for $\operatorname{Spec} R[[u]]$ and $D_R^\star$ for $\operatorname{Spec} R((u))$. Let X be an R-scheme and let $R \to S$ be a ring homomorphism. Write $X \otimes_R S$ for $X \times_{\operatorname{Spec} R} \operatorname{Spec} S$.

Let P be a $\mathcal{G}$-torsor over $D_R^\star$. Define the groupoid Gr_P over $\operatorname{Spec} R$ as follows. Let $R \to S$ be a ring homomorphism. Set $\mathrm{Gr}_P(S)$ to be the groupoid of pairs (T, γ) where T is a $\mathcal{G}$-torsor over D_S and γ is an isomorphism $T|_{D_S^\star} \xrightarrow{\sim} P \times_{D_R^\star} D_S^\star$. Note that when $P = \mathcal{G}_R := \mathcal{G} \otimes_{\mathcal{O}} R$ is the trivial $\mathcal{G}$-torsor, $\mathrm{Gr}_{\mathcal{G}_R}$ is the usual affine Grassmannian.

As a consequence of Lemma A.1.3, the groupoid Gr_P is stacky. Moreover, because of the framing, Gr_P is a setoid.

A.2.1. Lemma When $\mathcal{G} = \mathrm{GL}_N$, the groupoid Gr_P is representable by an ind-proper algebraic space.

Proof. It is [Dri06, Proposition 3.8]. □

A.2.2. Let $R \to S$ be a ring homomorphism. By unwinding the definitions, there is an isomorphism of groupoids $\mathrm{Gr}_{P \times_{D_R^\star} D_S^\star} \xrightarrow{\sim} \mathrm{Gr}_P \times_{\operatorname{Spec} R} \operatorname{Spec} S$.

Let $i : \mathcal{G} \to \mathrm{GL}_N$ be a closed immersion, then there is an obviously defined pushforward morphism $i_* : \mathrm{Gr}_P \to \mathrm{Gr}_{i_* P}$, where $i_* P := P \times^{\mathcal{G}} \mathrm{GL}_N$.

A.2.3. Lemma Assume $\mathcal{G}$ be a connected reductive group scheme over $\mathcal{O}$. Let $R \to S$ be a ring homomorphism. Then the map $|\mathrm{Gr}_P(S)| \to |\mathrm{Gr}_{i_*P}(S)|$ is injective.

Proof. Suppose $|\mathrm{Gr}_P(S)| \neq \emptyset$. Let $(T, \gamma) \in \mathrm{Gr}_P(S)$. Since $\mathcal{G}$ is smooth, there is an étale cover $\operatorname{Spec} S' \to \operatorname{Spec} S$ such that $T \times_{D_S} D_{S'}$ is a trivial $\mathcal{G}$-torsor over $D_{S'}$. Now that $\mathrm{Gr}_P \times_{\operatorname{Spec} R} \operatorname{Spec} S' \to \mathrm{Gr}_{i_*P} \times_{\operatorname{Spec} R} \operatorname{Spec} S'$ is relatively representable by a closed immersion (see, for example, [Lev13, Corollary 3.3.10]). By descent ([Stacks, Tag 04SK] and [Stacks, Tag 0420]), $\mathrm{Gr}_P \times_{\operatorname{Spec} R} \operatorname{Spec} S \to \mathrm{Gr}_{i_*P} \times_{\operatorname{Spec} R} \operatorname{Spec} S$ is also relatively representable by a closed immersion. Hence $|\mathrm{Gr}_P(S)| \to |\mathrm{Gr}_{i_*P}(S)|$ is injective. □

A.2.4. Lemma The morphism of groupoids $\mathrm{Gr}_P \to \mathrm{Gr}_{i_*P}$ is relatively representable by a closed immersion.

Proof. Fix $(M, \gamma_M) \in \mathrm{Gr}_{i_*P}(R)$. Write X for $\mathrm{Gr}_P \times_{\mathrm{Gr}_{i_*P}} \operatorname{Spec} R$. An object of $X(S)$ is a pair $((T, \gamma), \alpha)$ where $(T, \gamma) \in \mathrm{Gr}_P(S)$ and α is an isomorphism $i_*T \xrightarrow{\sim} M \otimes_{R[[u]]} S[[u]]$ which is compatible with γ and γ_M.

Define the groupoid Y over $R[[u]]$-algebras as follows: set $Y(B)$ to be the groupoid of pairs (T, α) where T is a $\mathcal{G}$-torsor over $\operatorname{Spec} B$ and α is an isomorphism $i_*T \xrightarrow{\sim} M \otimes_{R[[u]]} B$. Let $R \to S$ be a ring homomorphism. There is a map $X(S) \to Y(S[[u]])$ which sends $((T, \gamma), \alpha) \mapsto (T, \alpha)$. By Lemma A.2.3, the map $|X(S)| \to |Y(S[[u]])|$ is injective. It is standard that Y is represented by an affine scheme (namely, by $M \times^{\mathrm{GL}_N} (\mathrm{GL}_N / \mathcal{G})$). The image of $X(S)$ are objects (T, α) of $Y(S[[u]])$ whose restriction on $D_S^\star$ is $(P \times_{D_R^\star} D_S^\star, (P \to i_*P) \times_{D_R^\star} D_S^\star)$. By [Lev13, Lemma 3.3.9], there exists a closed subscheme $\operatorname{Spec} R/I \subset \operatorname{Spec} R$ such that $|X(S)|$ is non-empty if and only $R \to S$ factors through $\operatorname{Spec} R/I$. Hence $X \cong X \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$. By replacing R by R/I, we can assume $X(R) \neq \emptyset$. Now that there is an fppf ring map $R \to S$ such that $P \times_{D_R^\star} D_S^\star$ is a trivial $\mathcal{G}$-torsor; and $X \times_{\operatorname{Spec} R} \operatorname{Spec} S \cong \mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Gr}_{\mathrm{GL}_N}} \operatorname{Spec} S$, which is well-known to be a closed subscheme of $\operatorname{Spec} S$. This lemma is now proved by descent. □

A.2.5. Lemma Let X be a closed subscheme of the ind-scheme Gr_P. There is a Nisnevich cover $\operatorname{Spec} S \to \operatorname{Spec} R$ such that $X \times_{\operatorname{Spec} R} \operatorname{Spec} S$ is a projective scheme over $\operatorname{Spec} S$.

Proof. By Lemma A.2.4, it is reduced to the GL_N-case ([Dri06, Proposition 3.8]). □

B. Tannakian categories

An *exact category* is an additive category where a class of short exact sequences is specified. An *exact functor* is an additive functor which takes a short exact sequence to a short exact sequence.

A *monoidal category* is a tuple $(\mathcal{C}, -\otimes -, \mathbb{I})$ where $\mathcal{C}$ is a category, $-\otimes - : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is a bifunctor called the tensor product, and $\mathbb{I}$ is an object of $\mathcal{C}$ called the unit object. For a pair of objects (X, Y), the *internal Hom* $\underline{\mathrm{Hom}}(X, Y)$ (if it exists) is defined to be the object representing the functor $T \mapsto \mathrm{Hom}(T\otimes X, Y)$. For an object X, the *dual* $X^\vee$ (if it exists) is defined to be the object representing $\underline{\mathrm{Hom}}(X, \mathbb{I})$.

Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories. A *lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a pair (F, α) where F is a functor from $\mathcal{C}$ to $\mathcal{D}$ sending $\mathbb{I}$ to $\mathbb{I}$, and α is a natural transformation from the bifunctor $F(-)\otimes F(-)$ to the bifunctor $F(-\otimes -)$ satisfying some coherence conditions. F is said to be a *strict monoidal functor* if α is a natural isomorphism. If a strict monoidal functor has a right adjoint functor, then the right adjoint functor is canonically a lax monoidal functor.

Let $\mathcal{C}$ be an exact monoidal category. An object X is said to be an *invertible object* if the functor $-\otimes X$ is an exact equivalence.

Let $\mathcal{D}$, $\mathcal{E}$ be exact monoidal categories. Denote by $[\mathcal{D}, \mathcal{E}]^\otimes$ the category of faithful, exact, strict monoidal functors $\mathcal{D}\to\mathcal{E}$. A morphsim $\alpha : F \to G$ is a natural transformation satisfying $\alpha_{A\otimes B} = \alpha_A \otimes \alpha_B$ for $A, B \in \mathcal{D}$ and $\alpha_{\mathbb{I}} = \mathrm{id}$.

A *rigid category* is a monoidal category C such that (1) internal Hom always exists; (2) the morphism $\underline{\mathrm{Hom}}(X_1, Y_1)\otimes\underline{\mathrm{Hom}}(X_1, Y_2) \to \underline{\mathrm{Hom}}(X_1\otimes X_2, Y_1\otimes Y_2)$ is an isomorphism for objects X_1, Y_1, X_2 and Y_2; (3) the morphism $X \to (X^\vee)^\vee$ is an isomorphism. A strict monoidal functor between rigid categories automatically preserves inner Hom and duality [DM82, Proposition 1.9].

A monoidal category is said to be a *symmetric monoidal category* if it is equipped with a natural isomorphism $s_{A,B} : A\otimes B \to B\otimes A$ satisfying the obvious coherence conditions.

B.0.1. Let Z be a scheme. The category Coh_Z of coherent sheaves on Z is an abelian symmetric monoidal category. The category Vect_Z of finitely generated projective $\mathcal{O}_Z$-modules is an exact symmetric monoidal category. Let $f : Z \to Y$ be a proper morphism. The pullback functor $f^* : \mathrm{Coh}_Y \to \mathrm{Coh}_Z$ is a strict monoidal functor, and hence its right adjoint $f_* : \mathrm{Coh}_Z \to \mathrm{Coh}_Y$ is a lax monoidal functor. [1]

B.0.2. Setting and some monoidal categories Let R be a ring. Let Z be an R-scheme. Let $U \subset Z$ be an open dominant subscheme. Let $\varphi : Z \to Z$ be a morphism which map U to U.

- Define $\mathrm{Coh}_{Z,U,\varphi}$ to be the category of coherent $\mathcal{O}_Z$-modules $\mathcal{F}$, together with a homomorphism $\phi : \varphi^*\mathcal{F}|_U \to \mathcal{F}|_U$ (which we call a φ-structure).
- Define $\mathrm{Coh}^{\text{ét}}_{Z,U,\varphi}$ to be the full subcategory of $\mathrm{Coh}_{Z,U,\varphi}$ consisting of objects whose φ-structure is an isomorphism.
- Define $\mathrm{Vect}_{Z,U,\varphi}$ (resp. $\mathrm{Vect}^{\text{ét}}_{Z,U,\varphi}$) to be the full subcategory of $\mathrm{Coh}_{Z,U,\varphi}$ (resp. $\mathrm{Coh}^{\text{ét}}_{Z,U,\varphi}$) consisting of finitely generated projective $\mathcal{O}_Z$-modules. Note that $\mathrm{Vect}^{\text{ét,ef}}_{Z,U,\varphi}$ is also a full subcategory of $\mathrm{Vect}_{Z,\varphi}$.

We also call objects of $\mathrm{Coh}^{\text{ét}}_{Z,U,\varphi}$ modules with étale φ-structure.

[1] When $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a morphism of affine schemes, the lax monoidal structure of f_* is given by the map $M \otimes_A N \to M \otimes_B N$.

Lemma The category $\mathrm{Vect}^{\text{ét}}_{Z,U,\varphi}$ is an exact, rigid, symmetric monoidal category.

Proof. Since the inverse image functor φ^* is a strict monoidal functor, the tensor product of two étale ϕ-structure is an étale ϕ-structure. The unit object in $\mathrm{Vect}^{\text{ét}}_{Z,U,\varphi}$ is the structure sheaf $\mathcal{O}_Z$ together with the identification $\varphi^*\mathcal{O}_Z = \mathcal{O}_Z \otimes_{\varphi^{-1}\mathcal{O}_Z} \mathcal{O}_Z = \mathcal{O}_Z$.

Moreover, since the inverse image functor φ^* preserves the sheaf Hom, internal Homs in $\mathrm{Vect}^{\text{ét}}_{Z,U,\varphi}$ are representable. $\square$

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