ALGORITHMIC UNIVERSAL MOD p ÉTALE (φ, Γ)-MODULES

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Abstract. Let $F/\mathbb{Q}_p$ be a finite extension and let $\breve{G}$ be a connected reductive group with connected center satisfying $p > h_{\breve{G}}$. We write down explicit polynomial equations for the reduced Emerton-Gee stacks (the Borel and the twisted Borel versions that jointly cover the entire $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$), by introducing the *mod p Weil-Deligne stacks* capturing derived structures of (φ, Γ)-modules in reduced families.

This allows us to algorithmically compute the set of irreducible components of $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$, thereby establishing its equidimensionality, and the existence of crystalline lifts by showing the total number of irreducible components equals the number of (mod p) crystalline (or potentially semistable) components. The last step of the arguments is to interpolate the *rigid analytic Weil-Deligne stacks* and the mod p Weil-Deligne stacks to deduce the number of potentially semistable components.

An initial implementation of the algorithms based on Gröbner basis establishes the existence of crystalline lifts in the $\breve{G} = \mathrm{F}_4$ case for all $F/\mathbb{Q}_p$, and in the remaining $\breve{G} = \mathrm{E}_6, \mathrm{E}_7$, and E_8 cases for all but finitely many $F/\mathbb{Q}_p$.

Contents

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1. Introduction

Let $F/\mathbb{Q}_p$ be a finite extension and let $\breve{G}$ be a connected reductive group with connected center. The assumption that $p > h_{\breve{G}}$ is in force throughout the paper to enable the Baker-Campbell-Hausdorff formula and the truncated $\log_{\leqslant h_{\breve{G}}}$ and $\exp_{\leqslant h_{\breve{G}}}$.

We make major progress towards the open problem of the existence of crystalline lifts:

Theorem 1.1. *(Theorem 16.5) If $\breve{G} = \mathrm{F}_4$, then all $\mathrm{Gal}_F \to \breve{G}(\bar{\mathbb{F}}_p)$ have a crystalline lift.*

If $\breve{G}$ is an arbitrary reductive group and $[F : \mathbb{Q}_p] > \frac{\dim \breve{G}}{2}$, then all $\mathrm{Gal}_F \to \breve{G}(\bar{\mathbb{F}}_p)$ have a crystalline lift.

An algorithm (c.f. `github.com/mocham/AlgEG` for the implementation) is presented in Section 15 to resolve the remaining small field cases. Experiments show that a computationally efficient cone model suffices for our purposes, and thus the bottleneck is the very large number of parallelizable tasks rather than the computational cost of a single task. A complete resolution of this project seems realistic given abundant computing resources.

Our method is based on explicit polynomial equations for the reduced Emerton-Gee stacks. The structure of the polynomial equations yields the following purity result that is worth recording:

Theorem 1.2. *(Corollary 6.9) Write $\breve{B}$ for the Borel of $\breve{G}$. If $\dim \mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}} \leqslant [F : \mathbb{Q}_p] \dim \breve{G}/\breve{B}$, then $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}}$ is equidimensional of dimension $[F : \mathbb{Q}_p] \dim \breve{G}/\breve{B}$.*

As opposed to our earlier expectation, $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}}$ can have more irreducible components than $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$:

Theorem 1.3. *(Proposition 14.2, Corollary 15.8) Let $\breve{G} = \mathrm{F}_4$.*

- *If $F \neq \mathbb{Q}_p$, then $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}} \to \mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ induces a bijection of irreducible components.*
- *If $F = \mathbb{Q}_p$, then $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}}$ is still equidimensional but has 4 extra irreducible components compared to that of $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$.*

On the other hand, if $\breve{G}$ is an arbitrary reductive group and $[F : \mathbb{Q}_p] > \frac{\dim \breve{G}}{2}$, then $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}} \to \mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ induces a bijection of irreducible components.

Analysis of the moduli of Weil-Deligne representations in characteristic 0 (that dates back to Kisin [Kis08]) shows that the (reduced) special fiber of a crystalline stack of regular Hodge type is necessarily equidimensional of dimension $[F : \mathbb{Q}_p] \dim \breve{G}/\breve{B}$. Combining this with the purity result (c.f. Theorem

1.2), proving and disproving the existence of crystalline lifts both amount to counting the number of irreducible components.

The starting point of this paper is to mimic Kisin's approach of analyzing the generic fiber of the potentially semistable deformation rings using the Weil-Deligne moduli space, and to study the geometry of the reduced Emerton-Gee stacks by constructing a mod p version of Weil-Deligne stacks that parametrize pairs $(\bar{r}, \bar{N})$ where $\bar{r}$ is a semisimple Galois representation and $\bar{N}$ is a monodromy operator. These ideas are implicit in our previous work [Lin25], where we geometrize the Serre weights (a mod p notion) as irreducible components of the rigid analytic Borel-valued Weil-Deligne stacks.

Before we explain our methods, we briefly review the past works.

1.1. **Past works.**

1.1.1. [EG23] initiated the use of the reduced moduli stack of étale (φ, Γ)-modules to study the existence of crystalline lifts. They solved the GL_n-case by induction on n. Let $\bar{\rho} : \mathrm{Gal}_F \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be a mod p Galois representation and let $\bar{\alpha} : \mathrm{Gal}_F \to \mathrm{GL}_m(\bar{\mathbb{F}}_p)$ be an irreducible mod p Galois representation together with a fixed lift α, they proved the following criterion via ingenious Galois cohomology techniques:

> $(*)$ If the locus where $\dim \mathrm{Ext}^2(\bar{\alpha}, -) \geqslant n$ has codimension at least n in $\mathcal{X}^{\mathrm{EG}}_{\mathrm{GL}_n,\mathrm{red}}$, then each extension class $[\bar{c}] \in \mathrm{Ext}^1(\bar{\alpha}, \bar{\rho})$ is in the image of $\mathrm{Ext}^1(\alpha, \rho) \to \mathrm{Ext}^1(\bar{\alpha}, \bar{\rho})$ for a suitable crystalline lift ρ of $\bar{\rho}$.

The existence of crystalline lifts for GL_n is thus reduced to dimension counting: cut $\mathcal{X}^{\mathrm{EG}}_{\mathrm{GL}_n,\mathrm{red}}$ into many strata, and show the obstruction dimension is smaller than the codimension over each stratum.

For general reductive groups, we can try to adapt the strategy to an induction on Levi subgroups: For a parabolic subgroup $P = U^{\check{M}} \rtimes \check{M}$ where $U^{\check{M}}$ is the unipotent radical, we can try to lift extension classes $H^1(\mathrm{Gal}_F, U^{\check{M}}(\bar{\mathbb{F}}_p))$. The issue is that $U^{\check{M}}$ is a non-abelian group for other reductive groups, and $H^1(\mathrm{Gal}_F, U^{\check{M}}(\bar{\mathbb{F}}_p))$ lacks meaningful algebraic or geometric structures for a similar analysis. The even more serious issue is that this approach implicitly assumes that a P-valued mod p Galois representation always admits a P-valued crystalline lift. Even though we don't know of a counterexample yet, we see Theorem 1.3 as strong negative evidence for this — a generic $\bar{\mathbb{F}}_p$-point on the 4 extra irreducible components likely does not have a Borel-valued lift.

Remark 1.4. We did manage to extend the methods of [EG23] to the case of classical groups (c.f. [Lin25B]). The key is that $U^{\check{M}}$ is very close to being an abelian group when $\check{M}$ is a maximal proper Levi. The dimension counting process is much more complicated compared to the GL_n-case as each stratum is no longer an affine space but is rather an affine cone.

1.1.2. *Reflections on [EG23].* The main application of the existence of crystalline lifts in their work is the so-called *qualitative Breuil-Mézard* conjecture, which asserts that $\mathrm{CH}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{n,\mathrm{red}})$ is a free abelian group generated by the Serre weights.

In their work, $H^1(\mathrm{Gal}_F, U^{\check{M}}(\bar{\mathbb{F}}_p))$ is very structured while $\mathrm{CH}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{n,\mathrm{red}})$ seems to be a structure-less object. This explains why they exploit $H^1(\mathrm{Gal}_F, U^{\check{M}}(\bar{\mathbb{F}}_p))$ to calculate $\mathrm{CH}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{n,\mathrm{red}})$.

The situation changes for general reductive groups: $H^1(\mathrm{Gal}_F, U^{\check{M}}(\bar{\mathbb{F}}_p))$ is now completely structure-less. However, $\mathrm{CH}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\check{G},\mathrm{red}})$ is a structured object, through its connection to the (rigid analytic) Weil-Deligne stacks.

1.1.3. In his PhD thesis [Mu13], Muller constructs crystalline lifts for GL_2 and GL_3 using explicit unramified twists.

His method relies on strict genericity conditions on the relative positions of the Galois cohomology classes, which are not satisfied in full generality for GL_4. Nevertheless, it is powerful enough to show *sufficiently generic* Galois representations have a crystalline lift, which is, surprisingly, sufficient from the perspective of Theorem 1.2 — if $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}}$ is already known to be equidimensional and generic points of $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}}$ have crystalline lifts, then all points of $\mathcal{X}^{\mathrm{EG}}_{\breve{B},\mathrm{red}}$ have crystalline lifts. Indeed, in an early draft of this paper, we didn't know that an explicit presentation of the reduced Emerton-Gee stacks would exist, and we still managed to prove Theorem 1.2 using very indirect methods, and proceeded to establish the existence of crystalline lifts for F_4 using Muller's methods.

1.2. **The strategy.**

1.2.1. *Insights from rigid analytic geometry.* The story begins with our previous work on the rigid analytic Weil-Deligne stacks. We recall the following theorem:

Theorem 1.5. *([Lin25, Theorem 6]) Let* $\rho : \mathrm{Gal}_F \to \breve{M}(\bar{\mathbb{Q}}_p)$ *be a potentially semistable representation whose Hodge type is* P^-*-dominant. Endow* $\mathrm{Lie}\, U^{\breve{M}}(\bar{\mathbb{Q}}_p)$ *with* Gal_F*-action via* $\mathrm{ad}\rho$*.*

Any $\rho' : \mathrm{Gal}_F \to P(\bar{\mathbb{Q}}_p)$ *such that* $\rho' \times^P \breve{M} = \rho$ *is automatically potentially semistable of the same inertial type, with its possible Weil-Deligne types parameterized by* $H^2(\mathrm{Gal}_F, U^{\breve{M}}(\bar{\mathbb{Q}}_p))$.

This theorem suggests that if we write $\mathfrak{X}_{\breve{M}} = \mathfrak{X}^{\mathrm{pcrys},\underline{\lambda},\tau}_{\breve{M}}$ for the rigid analytic moduli stack of potentially crystalline Galois representations of Hodge type $\underline{\lambda}$ and inertial type τ, and write $\mathfrak{W}_P \to \mathfrak{X}_{\breve{M}}$ for the morphism relatively representing the groupoid

$$(\mathrm{Sp}\, A \to \mathfrak{X}_{\breve{M}}) \mapsto \{\text{Weil-Deligne representations } (r_A, N_A) \text{ whose } \breve{M}\text{-semisimplification is } D_{\mathrm{pst}}(V_A)\}$$

where D_{pst} is Fontaine's functor that sends a potentially semistable Galois representation (or a filtered (φ, N)-module) to a Weil-Deligne representation and V_A is the universal family over $\mathrm{Sp}\, A$, then fibers of $\mathfrak{W}_P \to \mathfrak{X}_{\breve{M}}$ are precisely the 2-cohomology $H^2(\mathrm{Gal}_F, U^{\breve{M}})$.

Moreover, if we write $\mathfrak{X}_P = \mathfrak{X}^{\mathrm{pst},\underline{\lambda},\tau}_P$ for the moduli of potentially semistable P-valued Galois representations, then there is a canonical commutative diagram

$$\begin{tikzcd} \mathfrak{X}_P \arrow[rr, "\mathrm{WD}"] \arrow[dr] & & \mathfrak{W}_P \arrow[dl] \\ & \mathfrak{X}_{\breve{M}} & \end{tikzcd}$$

where WD is the geometrization of the $D_{\mathrm{pst}}(V_A)$ functor. The morphism WD induces a bijection between irreducible components of $\mathfrak{X}_P$ and $\mathfrak{W}_P$.

Example 1.6. We consider the case where $\breve{G} = \mathrm{PGL}_2$ and $F = \mathbb{Q}_p$. Then $\breve{M} = \breve{T} = \mathbb{G}_m$. Suppose τ is the trivial inertial type and that $\underline{\lambda} = -1$ (the Barsotti-Tate case). Then $\mathfrak{X}_{\breve{M}} = [\mathfrak{G}_m/\mathfrak{G}_m]$ where $\mathfrak{G}_m = \mathrm{Sp}\, \mathbb{Q}_p\langle T^{\pm 1}\rangle$.

The morphism $\mathfrak{W}_P \to \mathcal{X}_{\breve{M}}$ is relatively representable by the Tor presheaf

$$\mathrm{Sp}\, A \mapsto \mathrm{Tor}_1^{\mathbb{Q}_p\langle T^{\pm 1}\rangle}(H^2(\mathrm{Gal}_F, U(\mathbb{Q}_p\langle T^{\pm 1}\rangle)), A)$$

where H^2 should really be interpreted as Herr complex cohomology. We have

$$H^2(\mathrm{Gal}_F, U(\mathbb{Q}_p\langle T^{\pm 1}\rangle)) = \mathbb{Q}_p\langle T^{\pm 1}\rangle/(T-1)$$

and thus $\mathfrak{W}_P \to \mathfrak{X}_{\check{M}}$ is relatively representable by

$$[\mathrm{Sp}\,\mathbb{Q}_p\langle T^{\pm 1}, Y\rangle/(Y(T-1))/\mathfrak{G}_m] \to [\mathrm{Sp}\,\mathbb{Q}_p\langle T^{\pm 1}\rangle/\mathfrak{G}_m].$$

Note that $\mathfrak{W}_P$ has two irreducible components:

- (the Steinberg component) $[\mathrm{Sp}\,\mathbb{Q}_p\langle Y\rangle/\mathfrak{G}_m]$, and
- (the non-Steinberg component) $[\mathrm{Sp}\,\mathbb{Q}_p\langle T^{\pm 1}\rangle/\mathfrak{G}_m]$.

On the other hand, the morphism

$$\mathfrak{X}_P \to \mathfrak{W}_P$$

relatively represents the connecting homomorphism δ of the long exact sequence:

$$H^1(\mathrm{Gal}_F, U(\mathbb{Q}_p\langle T^{\pm 1}\rangle)) \otimes_{\mathbb{Q}_p\langle T^{\pm 1}\rangle} A \to H^1(\mathrm{Gal}_F, U(A)) \xrightarrow{\delta} \mathrm{Tor}_1^{\mathbb{Q}_p\langle T^{\pm 1}\rangle}(H^2(M, \mathbb{Q}_p\langle T^{\pm 1}\rangle), A).$$

It is not hard to see $\delta = D_{\mathrm{pst}}$ coincides with Fontaine's functor.

The example above suggests that the Weil-Deligne construction has an alternative interpretation as inherent derived structures. This suggests that a mod p Weil-Deligne functor is feasible, at least for Borel-valued mod p Galois representations.

Notation 1.7. Write $\mathcal{X}_{\check{G}}^{\mathrm{EG}}$ for the Emerton-Gee stacks.
Write

$$\mathcal{X}_{\check{T}} := \mathcal{X}^{\mathrm{EG}}_{\check{T},\mathrm{red},\bar{\mathbb{F}}_p}$$

for the reduced torus-valued Emerton-Gee stack. Also write

$$\mathcal{X}_{\check{B}} := \mathcal{X}^{\mathrm{EG}}_{\check{B},\bar{\mathbb{F}}_p} \underset{\mathcal{X}^{\mathrm{EG}}_{\check{T},\bar{\mathbb{F}}_p}}{\times} \mathcal{X}_{\check{T}}$$

for the partially reduced Borel-valued Emerton-Gee stack.

1.2.2. *Gauge cochains.* Once we have obtained a mod p version of the Weil-Deligne stacks $\mathcal{W}_{\check{B}}$, we are able to exploit the morphism $\mathcal{X}_{\check{B}} \to \mathcal{W}_{\check{B}}$ to write down explicit polynomial presentations of $\mathcal{X}_{\check{B}}$ for the small-rank groups PGL_2 and PGL_3.

It is crucial that we are able to write down global equations for the small-rank groups, rather than just local equations — we do need to understand how various strata are glued in a precise manner.

After analysis of the explicit universal families for PGL_2 and PGL_3, we are able to recursively define a minimal set of 1-cochains in Herr complexes called the *gauge cochains* for PGL_n such that the universal family for $\mathcal{X}_{\check{B}_{\mathrm{PGL}_n}}$ can be expressed as a linear combination of the gauge cochains where the coefficients are valued in a multivariable Laurent polynomial ring $\bar{\mathbb{F}}_p[t_1^{\pm 1}, \dots]$. Such a free linear combination is not automatically a valid étale (φ, Γ)-module. Nevertheless, the structure of the gauge cochains makes it easy to write down the polynomial constraints imposed on the coefficients.

The purity theorem (c.f. Theorem 1.2) follows from Krull's Hauptidealsatz, once the polynomial equations are written down.

These PGL_n-constructions carry over to general reductive groups through the Baker-Campbell-Hausdorff formula and truncated logarithm maps.

1.2.3. *The cone models.* Directly using the polynomial equations to obtain the irreducible components of $\mathcal{X}_{\breve{B}}$ is not computationally viable. We make two optimizations:

- First, we can now divide $\mathcal{X}_{\breve{B}}$ into many small strata and only compute the irreducible components of these strata — thanks to the purity theorem.
- Second, over each stratum, we use the theory of Gröbner bases to prove that the dimension of the stratum will not drop after removing the cubic and higher degree terms.

It is a bit surprising that Gröbner basis, which is mostly used for concrete computations, can be utilized to prove abstract theorems.

The algebraic varieties (or stacks) defined by the truncated polynomial equations are called the *cone models*.

It turns out that the cone model of each stratum is of dimension strictly smaller than the expected dimension of $\mathcal{X}_{\breve{B}}$, unless the cone model is smooth and is *strictly isomorphic to the stratum itself*, for the cases covered by Theorem 1.1. We are thus able to compute the irreducible components of $\mathcal{X}_{\breve{B}}$.

1.2.4. *Unramified descent and the non-Borel valued mod p Galois representations.* We reduce the non-Borel-valued case to the Borel-valued case by unramified descent.

Indeed, if we want to write down global equations for open charts near a non-Borel-valued point, then tame descent is required. Fortunately, we only need equations for (non-open) strata covering the non-Borel-valued points, for which the unramified descent suffices.

1.3. **Final thoughts.** Many technical innovations are left out in the previous subsection, mainly because they are too specialized to be placed in this introduction.

During the preparation of this paper, we feel the notion of "mod p Weil-Deligne stacks" should have broader applications to the study of Galois deformation theory, integral p-adic Hodge theory, and mod p (derived) Hecke algebra.

Our ad hoc constructions can probably be explained and vastly generalized by the theory of prismatic F-crystals/gauges. We hope to come back to this in a follow-up project.

This paper is divided into three parts:

- **Part I** is the study of the geometry of mod p Borel-valued Emerton-Gee stacks,
- **Part II** is the study of the geometry of mod p twisted Borel-valued Emerton-Gee stacks, and
- **Part III** is the study of an integral version of Weil-Deligne stacks and the existence of crystalline lifts.

1.4. **Acknowledgements.** We thank Yu Min and Yupeng Wang for helpful discussions, especially Yu for explaining derived algebraic geometry to the author. We also thank Heejong Lee and Toby Gee for helpful correspondences.

1.5. **Use of AI.** The original manuscript and the original `github.com/mocham/AlgEG` implementation were written entirely by human.

Later, the author used DeepSeek Pro V4 and ChatGPT 5.6 Sol to fix typos and refractor the source code; the refractoring is done mainly by a DeepSeek agent and the verification/testing is done by a ChatGPT agent; the appendices are generated by a ChatGPT agent from the refractored source code.

We have instructed the AI to highlight all major changes colored as green, and we retain the coloring in the arXiv version.

Part I: The mod p Weil-Deligne stacks

2. Geometrization of the mod p local Euler characteristic

We have $\mathrm{Gal}(F_{\mu_{p^\infty}}/F) \cong \Delta_F \times \Gamma$, where Δ_F is the torsion factor and $\Gamma \cong \mathbb{Z}_p$. Set $F_{\mathrm{cyc}} = F(\mu_{p^\infty})^{\Delta_F}$. We have $\mathbf{E}_F = k_{F_{\mathrm{cyc}}}(\!(T_F)\!)$, where $k_{F_{\mathrm{cyc}}}$ is the residue field of F_{cyc}.

2.1. **Étale (φ, Γ)-modules with coefficients.** Let R be a commutative ring of finite type over $\bar{\mathbb{F}}_p$ and let $F/\mathbb{Q}_p$ be a finite extension. Following the construction of Emerton and Gee, we consider the period ring

$$\mathbf{E}_{F,R} := \left(\varprojlim_n (\mathbf{E}_F^+/T_F^n) \otimes_{\mathbb{F}_p} R \right) [1/T_F].$$

An étale (φ, Γ)-module over $\mathbf{E}_{F,R}$ is defined as a finite projective $\mathbf{E}_{F,R}$-module M equipped with a semilinear Γ-action and a semilinear φ-action such that the linearized map $\varphi^* M \to M$ is an isomorphism.

2.2. **Herr complexes.** For an étale (φ, Γ)-module M over $\mathbf{E}_{F,R}$, the Herr complex $C^\bullet_{\mathrm{Herr}}(M)$ is $0 \to M \xrightarrow{d^0_{\mathrm{Herr}}} M \oplus M \xrightarrow{d^1_{\mathrm{Herr}}} M \to 0$, where the maps are given by:

$$\begin{aligned} d^0_{\mathrm{Herr}}(x) &= ((\varphi_M - 1)x, (\gamma_M - 1)x) \\ d^1_{\mathrm{Herr}}(x, y) &= (\gamma_M - 1)x - (\varphi_M - 1)y \end{aligned}$$

For any algebra S of finite type over R, let $M_S = M \otimes_{\mathbf{E}_{F,R}} \mathbf{E}_{F,S}$. We define Z^i, B^i, and H^i as presheaves on the category of finite type R-algebras. Emphasizing the dependence on M, we write

$$\begin{aligned} Z^i(M, S) &:= Z^i(C^\bullet_{\mathrm{Herr}}(M_S)) =: Z^i_{\mathrm{Herr}}(M_S), \\ B^i(M, S) &:= B^i(C^\bullet_{\mathrm{Herr}}(M_S)) =: B^i_{\mathrm{Herr}}(M_S), \\ H^i(M, S) &:= H^i(C^\bullet_{\mathrm{Herr}}(M_S)) =: H^i_{\mathrm{Herr}}(M_S). \end{aligned}$$

We note that while $Z^i(M, -)$, $B^i(M, -)$, and $H^i(M, -)$ are generally only presheaves of $\mathcal{O}_{\mathrm{Spec}\, R}$-modules, with the exception of $H^2(M, -)$ being a coherent sheaf over $\mathrm{Spec}\, R$.

Remark 2.1. For the technical developments of this paper, all presheaves are understood as presheaves over *the big affine site*. We warn the reader that the quasi-coherent sheaves over the big affine site over $\mathrm{Spec}\, R$ do *not* form an abelian category, despite that they do form an abelian category over the small Zariski site over $\mathrm{Spec}\, R$. Kernels, cokernels and images are always taken over the big affine site.

We have the following observation:

Lemma 2.2. Suppose that R is a PID, and that M is an étale (φ, Γ)-module over $\mathrm{Spec}\, R$. There is a *surjective* homomorphism of presheaves of $\mathcal{O}_{\mathrm{Spec}\, R}$-modules:

$$H^1(M, S) \to \mathrm{Tor}_1^R(H^2(M, R), S).$$

Proof. By [EG23, Theorem 5.1.22], Herr cohomology is represented by a perfect complex $P^\bullet$ of finite projective R-modules representing Herr cohomology and satisfying, functorially in $R \to S$,

$$P^\bullet \otimes_R^{\mathbf{L}} S \simeq C^\bullet_{\mathrm{Herr}}(M_S).$$

Since R is a PID, the universal-coefficient spectral sequence degenerates to the short exact sequence

$$0 \to H^1(P^\bullet) \otimes_R S \to H^1(P^\bullet \otimes_R S) \to \mathrm{Tor}_1^R(H^2(P^\bullet), S) \to 0.$$

The base-change identification gives $H^i(P^\bullet) = H^i(M, R)$ and $H^i(P^\bullet \otimes_R S) = H^i(M, S)$, and hence the claimed surjection. □

The presheaf $\mathrm{Tor}_1^R(H^2(M, R), -)$ is often representable by a finite type affine scheme over $\operatorname{Spec} R$:

Lemma 2.3. Let $X = \operatorname{Spec} R$ be an affine scheme, and let F be a coherent sheaf over X of projective dimension $\leqslant 1$. Then $\mathrm{Tor}_1^R(F, -)$ is representable by a finite type scheme over $\operatorname{Spec} R$.

Proof. Since $\mathrm{pd}_R(F) \leqslant 1$, we have a projective resolution:

$$0 \to P_1 \xrightarrow{\phi} P_0 \to F \to 0.$$

For any R-algebra S, we tensor this sequence over R with S. Since P_0 is projective, $\mathrm{Tor}_1^R(P_0, S) = 0$, which yields the exact sequence:

$$0 \to \mathrm{Tor}_1^R(F, S) \to P_1 \otimes_R S \xrightarrow{\phi \otimes \mathrm{id}_S} P_0 \otimes_R S \to F \otimes_R S \to 0$$

From this, we can identify our functor exactly as the kernel of a map of projective modules:

$$\mathcal{F}(S) = \mathrm{Tor}_1^R(F, S) \cong \ker(P_1 \otimes_R S \to P_0 \otimes_R S).$$

We now show this kernel is representable. For any finitely generated projective R-module P, the functor $S \mapsto P \otimes_R S$ is represented by the affine group scheme $\mathbb{V}(P^*) = \operatorname{Spec}(\mathrm{Sym}_R(P^*))$, where $P^* = \mathrm{Hom}_R(P, R)$ is the dual module. Let $W_i = \operatorname{Spec}(\mathrm{Sym}_R(P_i^*))$ for $i = 0, 1$. The R-module homomorphism $\phi : P_1 \to P_0$ induces a natural transformation of functors $W_1 \to W_0$.

The presheaf $\mathcal{F}$ is the kernel of the map $W_1 \to W_0$ in the category of abelian group-valued functors. Categorically, the kernel of a morphism to a group object is the fiber product over the zero-section:

$$\mathcal{F} \cong W_1 \times_{W_0} \{0\}$$

where $\{0\} \cong \operatorname{Spec}(R)$ is the zero-section of the group scheme W_0. □

Example 2.4. Consider $X = \operatorname{Spec} \mathbb{F}_p[t]$, and let F be the skyscraper sheaf $R/(t)$ supported as $t = 0$.

Then F itself is *not* representable by finite type schemes (or algebraic spaces) over X. On the other hand, $\mathrm{Tor}_1(F, -)$ is representable by the union of two axis lines: $\operatorname{Spec} \mathbb{F}_p[t, s]/(ts)$. Indeed, it is the fiber at the zero section of the vector space scheme morphism

$$\operatorname{Spec} \mathbb{F}_p[t, s] \to \operatorname{Spec} \mathbb{F}_p[t, s], (t, s) \mapsto (t, ts).$$

Even though the coherent sheaf F is not representable by algebraic spaces. The *stacky version* of F is indeed representable by finite type algebraic stacks. Consider the perfect complex

$$[C^{-1} \to C^0] := [R \xrightarrow{t} R],$$

supported in degrees $-1, 0$. The *strict Picard stack* associated to the perfect complex $[C^{-1} \to C^0]$ is representable by the algebraic stack

$$[\operatorname{Spec} \mathbb{F}_p[t, s] / \operatorname{Spec} \mathbb{F}_p[s]]$$

where the action is given by $s \cdot (t, s') = (t, s' + ts)$. The two stacks $[\operatorname{Spec} \mathbb{F}_p[t,s]/\operatorname{Spec} \mathbb{F}_p[s]]$ and $\operatorname{Spec} \mathbb{F}_p[t,s]/(ts)$ are Koszul dual to each other.

Lemma 2.5. Suppose $R = \bar{\mathbb{F}}_p[t, t^{-1}]$ and let M be a rank 1 étale (φ,Γ)-module over R.

If $H^2_{\mathrm{Herr}}(M) \cong \prod_{i=1}^k R/(t - t_i)$ for $t_i \in \bar{\mathbb{F}}_p$, then $\operatorname{Tor}_1^R(H^2(M,R), -)$ is representable by a scheme obtained by gluing

$$\operatorname{Spec} \bar{\mathbb{F}}_p[t, t^{-1}, s_i]/(s_i(t - t_i))$$

over $\operatorname{Spec} \bar{\mathbb{F}}_p[t, t^{-1}]$.

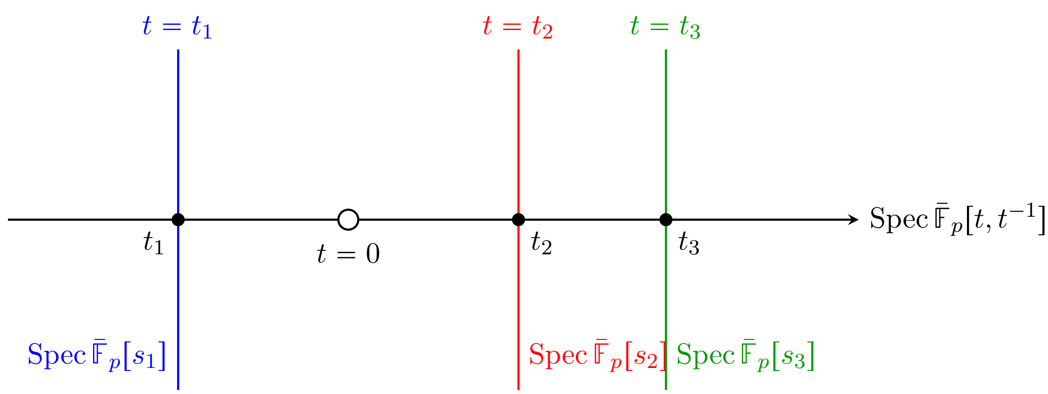


FIGURE 1. Illustration of $\operatorname{Tor}_1(H^2, -)$

Proof. This is Zariski local and, after reducing to $k = 1$, follows from Example 2.4. The resulting scheme is precisely the union of the axis lines depicted above. □

2.3. **Unramified extensions, and destackification.** By the local Euler characteristics, for Galois characters, we have $\dim H^1 = \dim H^0 + \dim H^2 + [F : \mathbb{Q}_p]$. Lemma 2.2 and Lemma 2.5 can be interpreted as a geometrization of the intuition that "H^1 contains H^2". In this subsection, we make precise the "H^1 contains H^0" counterpart.

Suppose $R = \bar{\mathbb{F}}_p[t, t^{-1}]$, and let M be a rank 1 étale (φ,Γ)-module over R that parametrizes *unramified* Galois characters. We have $\dim H^1_{\mathrm{Herr}}(\bar{\chi}_t, \bar{\mathbb{F}}_p) = \begin{cases} [F : \mathbb{Q}_p] + 1 & \bar{\chi}_t = 1 \\ [F : \mathbb{Q}_p] & \bar{\chi}_t \neq 1, \end{cases}$ where $\bar{\chi}_t$ is a $\bar{\mathbb{F}}_p$-fiber of M.

From the formula

$$d^0_{\mathrm{Herr}}(x) = ((\varphi_M - 1)x, (\gamma_M - 1)x),$$

we see that $B^1_{\mathrm{Herr}}(\bar{\chi}_t, \bar{\mathbb{F}}_p)$ contains the constant matrices $(\bar{\mathbb{F}}_p, 0)$ if and only if $\bar{\chi}_t \neq 1$. The upshot is that the constant matrices $(\bar{\mathbb{F}}_p, 0) \subset Z^1_{\mathrm{Herr}}$ are responsible for the dimension jump in H^0 and H^1, via the dimension drop in B^1, when $\bar{\chi}_t = 1$:

$$\begin{cases} (\bar{\mathbb{F}}_p, 0) = d^0_{\mathrm{Herr}}(\bar{\mathbb{F}}_p), & \bar{\chi}_t \neq 1 \\ (\bar{\mathbb{F}}_p, 0) \cap B^1_{\mathrm{Herr}} = 0, & \bar{\chi}_t = 1. \end{cases}$$

Definition 2.6. Let M be a rank 1 étale (φ, Γ)-module over $\bar{\mathbb{F}}_p[t, t^{-1}]$. We define the modified complex $C^{\bullet}_{\mathrm{Herr},+}(M, S)$ by setting:

$$\begin{aligned} C^0_{\mathrm{Herr},+}(M,S) &:= C^0_{\mathrm{Herr}}(M,S), \\ C^1_{\mathrm{Herr},+}(M,S) &:= (S \otimes_{\mathbb{F}_p} k_{F_{\mathrm{cyc}}}) \oplus C^1_{\mathrm{Herr}}(M,S), \\ C^2_{\mathrm{Herr},+}(M,S) &:= C^2_{\mathrm{Herr}}(M,S). \end{aligned}$$

The differentials are defined as follows. For degree 0, we set:

$$d^0_{\mathrm{Herr},+}(f) := \begin{cases} (\text{constant term of } f, d^0_{\mathrm{Herr}}(f)) & M \text{ is unramified} \\ (0, d^0_{\mathrm{Herr}}(f)) & M \text{ is ramified} \end{cases}$$

where the constant term is evaluated in $S \otimes_{\mathbb{F}_p} k_{F_{\mathrm{cyc}}}$. For degree 1, the differential is given by $d^1_{\mathrm{Herr},+}(s, x) := d^1_{\mathrm{Herr}}(x)$.

Here, the "constant term of f" depends on the choice of a basis element $v \in M_S^{\Gamma} \cong S \otimes_{\mathbb{F}_p} k_{F_{\mathrm{cyc}}}$.

Lemma 2.7. Let M be any rank unramified 1 étale (φ, Γ)-module over $R = \bar{\mathbb{F}}_p[t^{\pm 1}]$. We have $M^{\Gamma} = \bar{\mathbb{F}}_p[t^{\pm 1}] \otimes_{\mathbb{F}_p} k_{F_{\mathrm{cyc}}}$.

Proof. M^{Γ} is a quasi-coherent sheaf whose $\bar{\mathbb{F}}_p$-fibers are all isomorphic to $\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_p} k_{F'_{\mathrm{cyc}}}$. So, $M^{\Gamma} \cong R \otimes_{\mathbb{F}_p} k_{F'_{\mathrm{cyc}}}$ is finite free by Swan's theorem. □

Lemma 2.8. Then $[C^{\bullet}_{\mathrm{Herr},+}(M,S)]$ is a perfect complex with $H^0_{\mathrm{Herr},+}(M,S) = 0$ and $H^2_{\mathrm{Herr},+}(M,S) = H^2_{\mathrm{Herr}}(M,S)$.

Moreover, $\ker(H^1_{\mathrm{Herr},+}(M,S) \to \mathrm{Tor}_1^R(H^2_{\mathrm{Herr}}(M,R), S))$ is a finite free coherent sheaf of constant rank $([F : \mathbb{Q}_p] + [k_{F_{\mathrm{cyc}}} : \mathbb{F}_p])$.

Proof. The perfectness of $C^{\bullet}_{\mathrm{Herr},+}$ follows from the perfectness of $C^{\bullet}_{\mathrm{Herr}}$. It follows from the construction that $H^0_{\mathrm{Herr},+}(M,S) = 0$ and $H^2_{\mathrm{Herr},+}(M,S) = H^2_{\mathrm{Herr}}(M,S)$. The rest of the lemma follows from local Euler characteristic. □

The stackiness of the Borel-valued Emerton-Gee stacks originates from the H^0 terms. By using the modified complex $[C^{\bullet}_{\mathrm{Herr},+}]$, we can uniformly eliminate this stackiness.

3. Destackification and coordinates

Let $(G, B, T, \{X_\alpha\}_{\alpha \in \Delta})$ be a split pinned reductive group over F, and let $(\breve{G}, \breve{B}, \breve{T}, \{Y_\alpha\}_{\alpha \in \Delta})$ be the dual pinned group of G. Assume $p > h_{\breve{G}}$, where $h_{\breve{G}}$ is the Coxeter number of $\breve{G}$. In particular, the truncated log induces an isomorphism

$$\log := \log_{\leqslant h_{\breve{G}}} : U \xrightarrow{\cong} \mathrm{Lie}(U)$$

of schemes over $\bar{\mathbb{F}}_p$.

Let $U \subset \breve{B}$ be the unipotent radical and consider the *descending central series*

$$\mathrm{Lie}\, U_k = \begin{cases} \mathrm{Lie}\, U & k = 1 \\ [\mathrm{Lie}\, U, \mathrm{Lie}\, U_{k-1}] & k > 1. \end{cases}$$

and let $U_k \subset U$ be the corresponding connected subgroup. Set $\bar{U}_k := U_k/U_{k+1}$. Because p is a good prime for $\breve{G}$, the descending central filtration coincides with the height filtration, and we have

$$\bar{U}_k = \prod_{\alpha\in\Phi(B,T),\mathrm{ht}(\alpha)=k} U_{\alpha^\vee}\,,$$

where $U_{\alpha^\vee} \subset \breve{G}$ is the root group of root $\alpha^\vee \in \Phi(\breve{B},\breve{T})$. We remind the reader that $\mathrm{ht}(\sum_{\alpha\in\Delta} n_\alpha \alpha) := \sum n_\alpha \in \mathbb{Z}$, and that any positive root can be written as a sum $\alpha_1 + \cdots + \alpha_s$ with each $\alpha_i \in \Delta$ such that any partial sum $\alpha_1 + \cdots + \alpha_k$ is still a positive root in $\Phi(B,T)$ (c.f. [Hum72, Lemma 10.2.B]). A consequence of the partial sum property is that

$$[\mathrm{Lie}\,\bar{U}_1, \mathrm{Lie}\,\bar{U}_k] = \mathrm{Lie}\,\bar{U}_{k+1}. \tag{1}$$

Definition 3.1. The k-th truncated Borel-valued Emerton-Gee stack for G is, by definition,

$$\mathrm{tr}_k\,\mathcal{X}_{\breve{B}} := \mathcal{X}_{U/U_{k+1}\rtimes\breve{T}} := \mathcal{X}^{\mathrm{EG}}_{U/U_{k+1}\rtimes\breve{T}} \times_{\mathcal{X}^{\mathrm{EG}}_{\breve{T}}} \mathcal{X}^{\mathrm{EG}}_{\breve{T},\mathrm{red}}.$$

3.1. **Destackification.** For many technical applications, working with schemes or algebraic spaces is preferable to working with algebraic stacks. To facilitate this, we recursively construct a destackification $X_{\breve{B}} \to \mathcal{X}_{\breve{B}}$.

First, we fix a total ordering "$\leqslant$" of all positive roots $\Phi(\breve{B},\breve{T})^+$ that is compatible with heights in the sense that

$$\alpha \leqslant \beta \Rightarrow \mathrm{ht}(\alpha) \leqslant \mathrm{ht}(\beta).[1]$$

For each $\alpha \in \Phi(\breve{B},\breve{T})^+$ of height h, the notation below is understood in the quotient U/U_{h+1}. More precisely, we define the successive subquotients

$$U_{\leqslant\alpha} := \frac{U/U_{h+1}}{\prod_{\substack{\beta>\alpha\\ \mathrm{ht}(\beta)=h}} U_{\beta^\vee}}, \qquad U_{<\alpha} := \frac{U/U_{h+1}}{\prod_{\substack{\beta\geqslant\alpha\\ \mathrm{ht}(\beta)=h}} U_{\beta^\vee}}.$$

The height-h root groups are central in U/U_{h+1}; hence these quotients are well-defined and fit into an exact sequence

$$1 \to U_{\alpha^\vee} \to U_{\leqslant\alpha} \to U_{<\alpha} \to 1.$$

Thus all root products appearing below are regarded as the corresponding subquotients. For ease of notation, set $\mathcal{X}_{\leqslant\alpha} := \mathcal{X}_{U_{\leqslant\alpha}\rtimes\breve{T}} := \mathcal{X}^{\mathrm{EG}}_{U_{\leqslant\alpha}\rtimes\breve{T}} \times_{\mathcal{X}^{\mathrm{EG}}_{\breve{T}}} \mathcal{X}_{\breve{T},\mathrm{red}}$.

Our method of destackification is a natural geometric extension of Definition 2.6. Let $\alpha \in \Phi(\breve{B},\breve{T})^+$. Let $\breve{T} \to \mathbb{G}_{m,\alpha} \cong \mathbb{G}_m$ be the morphism corresponding to the root α. Note that the unramified locus of $\mathcal{X}_{\mathbb{G}_{m,\alpha}}$ is a connected component. Over the ramified locus $\mathcal{X}_{\leqslant\alpha} \to \mathcal{X}_{\mathbb{G}_{m,\alpha}}$, we simply define $\mathcal{X}^+_{\leqslant\alpha} := \mathcal{X}_{\leqslant\alpha} \times \mathbb{G}_a^{[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$. Write $\mathcal{X}_{\leqslant\alpha,\mathrm{un}} \subset \mathcal{X}_{\leqslant\alpha}$ be the open and closed substack that projects onto the unramified locus of $\mathcal{X}_{\leqslant\alpha}$. Over the unramified locus, we construct a $\mathbb{G}_a$-torsor

$$\mathcal{X}^+_{\leqslant\alpha,\mathrm{un}} \to \mathcal{X}_{\leqslant\alpha,\mathrm{un}}$$

as follows: Consider a morphism $\mathrm{Spec}\,A \to \mathcal{X}_{\leqslant\alpha,\mathrm{un}}$ corresponding to an étale (φ,Γ)-structure on a $(U_{\leqslant\alpha} \rtimes \breve{T})$-torsor P'_A over $\mathrm{Spec}\,\mathbf{E}_{F,A}$, and set $P_A := P'_A \times^{U_{\leqslant\alpha}\rtimes\breve{T}} (U_{<\alpha} \rtimes \breve{T})$. The objects of the fiber product $\mathcal{X}^+_{\leqslant\alpha,\mathrm{un}} \times_{\mathcal{X}_{\leqslant\alpha,\mathrm{un}}} \mathrm{Spec}\,A$ are pairs (P'_A, ξ), where:

[1]We apologize for the non-standard notation as $\alpha < \beta$ usually means $(\beta - \alpha)$ is dominant in the literature.

- P'_A is a $(U_{\leqslant\alpha} \rtimes \breve{T})$-torsor over $\operatorname{Spec} \mathbf{E}_{F,A}$ equipped with an identification $P'_A \times^{U_{\leqslant\alpha}\rtimes\breve{T}} (U_{<\alpha} \rtimes \breve{T}) = P_A$.
- $\xi : P_A \to P'_A$ is a Γ-stable section of the natural projection $P'_A \to P_A$. Here, a Γ-stable section is the geometrization of the "constant term coefficient" in Definition 2.6.

Relation to the modified Herr complex. For a rank-1 module M over $\bar{\mathbb{F}}_p[t^{\pm 1}]$ that is the universal unramified twist of a Galois character, an element $x \in M$ expands as $x = \sum_{n\in\mathbb{Z}} a_n T_F^n$, after the choice of a Γ-stable generator. The constant term $\operatorname{ct}(x) = a_0$ is the coefficient of T_F^0. A section ξ as above corresponds to choosing a Γ-stable element whose constant term is prescribed.

Lemma 3.2. *We have* $[\mathcal{X}^+_{\leqslant\alpha}/\mathbb{G}_a^{\oplus[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}] \cong \mathcal{X}_{\leqslant\alpha}$.

Proof. By construction $\mathcal{X}^+_{\leqslant\alpha}$ is a pseudo $\mathbb{G}_a^{\oplus[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$-torsor over $\mathcal{X}_{\leqslant\alpha}$. We still need to show that it is a genuine $\mathbb{G}_a^{\oplus[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$-torsor over the unramified locus. This is equivalent to the statement that all $P'_A \to P_A$ admits a Γ-stable section fppf locally in $\operatorname{Spec} A$. Since P_A is affine and $\operatorname{Res}_{\mathbf{E}_{F,A}/A} \mathbb{G}_a$ is quasi-coherent, the coherent cohomology $H^1(P_A, \operatorname{Res}_{\mathbf{E}_{F,A}/A} \mathbb{G}_a) = 0$, and P'_A is indeed a trivial $\operatorname{Res}_{\mathbf{E}_{F,A}/A} \mathbb{G}_a$-torsor over P_A.

Fix a section ξ_0, and consider the set of all sections $\{\xi\}$. The set $\{\xi - \xi_0\}$ of all sections is an étale (φ, Γ)-module M_0 over $\mathbf{E}_{F,A}$, and reducing to the universal case $A = \bar{\mathbb{F}}_p[t^{\pm 1}]$, M_0 is the unramified twist of a rank-1 (φ, Γ)-module, which always possesses a Γ-invariant generator. □

Definition 3.3. We recursively define destackifications $X_{\leqslant\alpha}$ for $\alpha \in \Phi^+$.

Since $\mathcal{X}_{\breve{T}} = \coprod[\breve{T}/\breve{T}]$, set $X_{<\alpha_{\min}} = X_{\breve{T}} := \coprod \breve{T}$, where the disconnected union is indexed over the set of Serre weights for $\mathrm{GL}_1(\kappa_F)$.

Assuming we have already constructed $X_{<\alpha}$, we define the next stage as:

$$X_{\leqslant\alpha} := X_{<\alpha} \times_{\mathcal{X}_{<\alpha}} \mathcal{X}^+_{\leqslant\alpha}.$$

Finally, setting $X_{\breve{B}} := X_{\leqslant\alpha_{\max}}$.

Proposition 3.4. $X_{\breve{B}} \to \mathcal{X}_{\breve{B}}$ *is smooth, surjective, affine, and all its fibers are isomorphic to* $\operatorname{Res}_{k_{F_{\mathrm{cyc}}}/\mathbb{F}_p} U \rtimes \breve{T}$.

Moreover, the formation of $X_{U_{\leqslant\alpha}\rtimes\breve{T}}$ *tautological and does not depend on the choice of the total ordering on* $\Phi(\breve{B}, \breve{T})$.

Proof. The fiber description follows inductively from the recursive construction: at each step we add a $\mathbb{G}_a^{\oplus[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$-torsor for the current root, and root groups at the same height commute, so the product is a torsor for the direct sum, giving the claimed fiber $\operatorname{Res}_{k_{F_{\mathrm{cyc}}}/\mathbb{F}_p} U \rtimes \breve{T}$. The "moreover" part follows because if α_1 and α_2 have the same height, then $U_{\alpha_1^\vee}$ commutes with $U_{\alpha_2^\vee}$. □

3.2. $\breve{T}$-stable normal subgroups.

If an algebraic subgroup K of U is stable under the $\breve{T}$-adjoint action, then $K = \prod_{\alpha\in\Phi_K\subset\Phi^+} U_{\alpha^\vee}$ is a product of root groups. The group K is normal if and only if

$$\alpha \in \Phi_K,\ \beta \in \Phi^+,\ \alpha + \beta \in \Phi^+ \Rightarrow \alpha + \beta \in \Phi_K.$$

Definition 3.5. Consider the descending central series $\operatorname{Lie} K_1 := \operatorname{Lie} K$, $\operatorname{Lie} K_h := [\operatorname{Lie} K, \operatorname{Lie} K_{h-1}]$. Set $\bar{K}_h := K_h/K_{h+1}$. Define the K-height so that $\mathrm{ht}_K(\alpha) = h \Leftrightarrow U_{\alpha^\vee} \subset \bar{K}_h$:

$$\bar{K}_h = \prod_{\mathrm{ht}_K(\alpha)=h} U_{\alpha^\vee}.$$

Write

$$\Delta_K := \{\alpha \in \Phi_K | \operatorname{ht}_K(\alpha) = 1\}$$

and call them the K-*simple roots*. Note that ht_K need not be additive, unless K is the unipotent radical of a parabolic.

3.3. **Coordinates.**

Definition 3.6. If $S \subset \Phi^+$ is a subset, write

$$U_S := \prod_{\alpha \in S} U_{\alpha^\vee},$$

for some ordering on S, and is treated as a subquotient of U when possible.

Fix a $\breve{T}$-stable normal subgroup $K = U_{\Phi_K} \subset U$. We fix a total ordering "$\underset{K}{\leqslant}$" on Φ_K compatible with the K-height. For $\alpha, \beta \in \Phi_K$, we write

$$\alpha \underset{K}{>} \beta$$

to indicate α is greater than β in the fixed total ordering. Write

$$\begin{aligned} \Phi_{K,\leqslant\alpha} &:= \{\beta \in \Phi_K : \beta \underset{K}{\leqslant} \alpha\}, \\ K_{\leqslant\alpha} &:= U_{\Phi_{K,\leqslant\alpha}}, \\ \operatorname{tr}_h K &:= U_{\Phi_{K,\operatorname{ht}_K \leqslant h}}. \end{aligned}$$

Definition 3.7. We fix a standard Chevalley basis $\{e_\eta\}_{\eta\in\Phi}$ for $\operatorname{Lie}(\breve{G})$. The Lie bracket satisfies

$$[e_\beta, e_\delta] = N_{\beta,\delta} e_\alpha \tag{2}$$

whenever $\beta + \delta = \alpha \in \Phi$, where $N_{\beta,\delta} \in \mathbb{Z}$ are the structure constants of the root system.

Definition 3.8. Let A be an $\bar{\mathbb{F}}_p$-algebra. Fix an étale (φ, Γ)-module

$$f_{<\alpha}, g_{<\alpha} \in (K_{<\alpha} \rtimes \breve{T})(\mathbf{E}_{F,A})$$

satisfying

$$f_{<\alpha}\varphi(g_{<\alpha}) = g_{<\alpha}\gamma(f_{<\alpha}).$$

Write

$$f_{<\alpha} = c_{f,<\alpha} f^{\mathrm{ss}}, \qquad g_{<\alpha} = c_{g,<\alpha} g^{\mathrm{ss}}$$

where

$$c_{f,<\alpha}, c_{g,<\alpha} \in K_{<\alpha}(\mathbf{E}_{F,A}), \qquad f^{\mathrm{ss}}, g^{\mathrm{ss}} \in \breve{T}(\mathbf{E}_{F,A}).$$

Write

$$\log_{\leqslant h_{\breve{G}}}(c_{f,<\alpha}) = \sum_{\beta \underset{K}{<} \alpha} c_{f,\beta}, \qquad \log_{\leqslant h_{\breve{G}}}(c_{g,<\alpha}) = \sum_{\beta \underset{K}{<} \alpha} c_{g,\beta}$$

where $c_{f,\beta}, c_{g,\beta} \in U_{\beta^\vee}(\mathbf{E}_{F,A})$. If α is the smallest root of height $(h+1)$, we also write

$$(\operatorname{tr}_h c_f, \operatorname{tr}_h c_g) = (c_{f,<\alpha}, c_{g,<\alpha}).$$

Theorem 3.9. Choose elements $c_{f,\alpha}, c_{g,\alpha} \in U_{\alpha^\vee}(\mathbf{E}_{F,A})$. Set

$$c_{f,\leqslant\alpha} := \exp_{\leqslant h_{\check{G}}}(c_{f,<\alpha} + c_{f,\alpha}), \qquad c_{g,\leqslant\alpha} := \exp_{\leqslant h_{\check{G}}}(c_{g,<\alpha} + c_{g,\alpha})$$

Then

$$(f_{\leqslant\alpha}, g_{\leqslant\alpha}) := (c_{f,\leqslant\alpha} f^{\mathrm{ss}}, c_{g,\leqslant\alpha} g^{\mathrm{ss}})$$

is an étale (φ, Γ)-module if and only if

$$(c_{f,\alpha} - g^{\mathrm{ss}}\gamma(c_{f,\alpha})g^{\mathrm{ss}-1}) - (c_{g,\alpha} - f^{\mathrm{ss}}\varphi(c_{g,\alpha})f^{\mathrm{ss}-1}) = D_\alpha(c_{f,<\alpha}, c_{g,<\alpha}),$$

where

$$D_\alpha(c_{f,<\alpha}, c_{g,<\alpha}) = \sum_{n=1}^{h_{\check{G}}} \sum_{\substack{\beta_1,\dots,\beta_n \in \Phi_K \\ \beta_i<\alpha,\ \beta_1+\cdots+\beta_n=\alpha}} \frac{1}{n!}[c_{g,\beta_1}, [c_{g,\beta_2}, \dots [c_{g,\beta_{n-1}}, g^{\mathrm{ss}}\gamma(c_{f,\beta_n})g^{\mathrm{ss}-1}]]]$$
$$- \sum_{n=1}^{h_{\check{G}}} \sum_{\substack{\beta_1,\dots,\beta_n \in \Phi_K \\ \beta_i<\alpha,\ \beta_1+\cdots+\beta_n=\alpha}} \frac{1}{n!}[c_{f,\beta_1}, [c_{f,\beta_2}, \dots [c_{f,\beta_{n-1}}, f^{\mathrm{ss}}\varphi(c_{g,\beta_n})f^{\mathrm{ss}-1}]]] \tag{3}$$

where the sums range over ordered tuples of positive roots, repetitions are allowed, and for $n = 1$ the nested bracket denotes its innermost term.

Proof. Note that $U_{\alpha^\vee}$ is contained in the center of $K_{\leqslant\alpha}$. Write the cocycle equation in logarithmic coordinates and move the two terms involving $c_{f,\alpha}$ and $c_{g,\alpha}$ to the left. For the remaining terms, use

$$\mathrm{Ad}(\exp x)(y) = \exp(\mathrm{ad} x)(y) = \sum_{m\geqslant 0} \frac{1}{m!}(\mathrm{ad} x)^m(y),$$

which is often called the Hadamard's Lemma. The root grading implies that the projection to $U_{\alpha^\vee}$ receives a contribution precisely from ordered tuples of positive roots whose sum is α; all longer terms vanish because every root occurring here is positive and has height at most $h_{\check{G}}$. Expanding the two conjugations gives the two nested-commutator sums in Equation 3, with opposite signs. The remaining linear term is $-d^1_{\mathrm{Herr}}(c_{f,\alpha}, c_{g,\alpha})$, which proves the equivalence. $\square$

Remark 3.10. For ease of notation, write $\gamma_g(u) := g^{\mathrm{ss}}\gamma(u)g^{\mathrm{ss}-1}$ and $\varphi_f(u) := f^{\mathrm{ss}}\varphi(u)f^{\mathrm{ss}-1}$.

Here are some basic observations:

$$(c_{f,\alpha} - \gamma_g(c_{f,\alpha})) - (c_{g,\alpha} - \varphi_f(c_{g,\alpha})) = -d^1_{\mathrm{Herr}}(c_{f,\alpha}, c_{g,\alpha})$$

is the usual differential in Herr complex; the usual cup product in Herr complexes is given by

$$\left(\frac{c_{f,\beta_1}}{e_{\beta_1}}, \frac{c_{g,\beta_1}}{e_{\beta_1}}\right) \cup \left(\frac{c_{f,\beta_2}}{e_{\beta_2}}, \frac{c_{g,\beta_2}}{e_{\beta_2}}\right) = \frac{c_{f,\beta_1}}{e_{\beta_1}}\varphi_f\left(\frac{c_{g,\beta_2}}{e_{\beta_2}}\right) - \frac{c_{g,\beta_1}}{e_{\beta_1}}\gamma_g\left(\frac{c_{f,\beta_2}}{e_{\beta_2}}\right)$$

and thus

$$[c_{g,\beta_1}, \gamma_g(c_{f,\beta_2})] - [c_{f,\beta_1}, \varphi_f(c_{g,\beta_2})] = -N_{\beta_1,\beta_2}\left(\frac{c_{f,\beta_1}}{e_{\beta_1}}, \frac{c_{g,\beta_1}}{e_{\beta_1}}\right) \cup \left(\frac{c_{f,\beta_2}}{e_{\beta_2}}, \frac{c_{g,\beta_2}}{e_{\beta_2}}\right) e_{\beta_1+\beta_2}$$

where N_{β_1,β_2} is the structural constant.

Definition 3.11. Write

$$c_\beta = (c_{f,\beta}, c_{g,\beta}) \in C^1_{\mathrm{Herr}}(U_{\beta^\vee}(\mathbf{E}_F)),$$

and

$$c_{\beta_1} \cup c_{\beta_2} := (\frac{c_{f,\beta_1}}{e_{\beta_1}}, \frac{c_{g,\beta_1}}{e_{\beta_1}}) \cup (\frac{c_{f,\beta_2}}{e_{\beta_2}}, \frac{c_{g,\beta_2}}{e_{\beta_2}}) e_{\beta_1+\beta_2}.$$

We also define the *higher cup product*

(4)

$$\begin{aligned}[c_{\beta_1}, c_{\beta_2}, \dots, c_{\beta_n}] :=& [c_{g,\beta_1}, [c_{g,\beta_2}, \dots [c_{g,\beta_{n-1}}, g^{\mathrm{ss}}\gamma(c_{f,\beta_n})g^{\mathrm{ss}-1}]]] - [c_{f,\beta_1}, [c_{f,\beta_2}, \dots [c_{f,\beta_{n-1}}, f^{\mathrm{ss}}\varphi(c_{g,\beta_n})f^{\mathrm{ss}-1}]]] \\ &\in C^2_{\mathrm{Herr}}([U_{\beta_1^\vee}, [U_{\beta_2^\vee}, \dots, [U_{\beta_{n-1}^\vee}, U_{\beta_n^\vee}]]](\mathbf{E}_{F,A})).\end{aligned}$$

So, $c_{\beta_1} \cup c_{\beta_2} = -\frac{1}{N_{\beta_1,\beta_2}}[c_{\beta_1}, c_{\beta_2}]$.

Corollary 3.12. If $(\mathrm{tr}_h c_f, \mathrm{tr}_h c_g)$ is an étale (φ,Γ)-module valued in $\mathrm{tr}_h K \rtimes \check{T}$, then it can be extended to $\mathrm{tr}_{h+1} K \rtimes \check{T}$ if and only if

$$\frac{1}{2} \sum_{\delta\in\Delta_K, \delta+\beta=\alpha, \mathrm{ht}_K(\beta)=h, \mathrm{ht}_K(\alpha)=h+1} N_{\delta,\beta} c_\delta \cup c_\beta \in \sum_{\mathrm{ht}_K(\alpha)=h+1} D_\alpha(\mathrm{tr}_{h-1} c_f, \mathrm{tr}_{h-1} c_g) + B^2_{\mathrm{Herr}}(\bar{K}_{h+1}(\mathbf{E}_{F,A})).$$

Proof. Clear from the definitions. □

Corollary 3.13. Suppose $\mathrm{ht}_K(\alpha) = h+1$. If $(c_{f,<\alpha}, c_{g,<\alpha})$ is an étale (φ,Γ)-module valued in $K_{<\alpha} \rtimes \check{T}$, then it can be extended to $K_{\leqslant\alpha} \rtimes \check{T}$ if and only if

$$\frac{1}{2} \sum_{\substack{\delta\in\Delta_K,\ \beta\in\Phi_{K,\mathrm{ht}_K=h} \\ \delta+\beta=\alpha}} N_{\delta,\beta} c_\delta \cup c_\beta \in D_\alpha(\mathrm{tr}_{h-1} c_f, \mathrm{tr}_{h-1} c_g) + B^2_{\mathrm{Herr}}(U_{\alpha^\vee}(\mathbf{E}_{F,A})).$$

Proof. Clear from the definitions. □

4. Example: PGL_2

4.1. **The Weil-Deligne stack and the equations.** The Borel subgroup $B = B_{\mathrm{PGL}_2}$ of PGL_2 is isomorphic to $\mathbb{G}_a \rtimes \mathbb{G}_m = \{\begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix}\} \subset \mathrm{GL}_2$.

Lemma 4.1. The morphism $\mathcal{X}_B \to \mathcal{X}_{\mathbb{G}_m}$ is relatively representable by the groupoid $[Z^1_{\mathrm{Herr}}/C^0_{\mathrm{Herr}}]$. Here, $[C^0_{\mathrm{Herr}} \to C^1_{\mathrm{Herr}} \to C^2_{\mathrm{Herr}}]$ is the Herr complex attached to the universal family over $\mathcal{X}_{\mathbb{G}_m}$.

Proof. Unravel the definitions. □

We remark that $\mathcal{X}_B \to \mathcal{X}_{\mathbb{G}_m}$ is *not* relatively representable by a strict Picard stack as Z^1_{Herr} is not a coherent sheaf.

Definition 4.2. Denote by $\mathcal{W}_B$ for the algebraic stack over $\mathcal{X}_{\mathbb{G}_m}$ that relatively represents the presheaf $\mathrm{Tor}_1^{\mathcal{O}_{X_{\mathbb{G}_m}}}(H^2_{\mathrm{Herr}}(\mathbf{E}_{F,\mathcal{O}_{X_{\mathbb{G}_m}}}), R)$ for each morphism $\mathrm{Spec}\, R \to \mathcal{X}_{\mathbb{G}_m}$. Set $W_B := \mathcal{W}_B \times_{\mathcal{X}_{\mathbb{G}_m}} X_{\mathbb{G}_m}$.

Definition 4.3. Let $C \subset \mathcal{W}_B$ be an irreducible component. We say C is a *Steinberg* component if its image $\bar{C}$ in $\mathcal{X}_{\mathbb{G}_m}$ is not an irreducible component. Set

$$\mathcal{X}_B(C) := C \times_{\mathcal{W}_B} \mathcal{X}_B,$$
$$X_B(C) := C \times_{\mathcal{W}_B} X_B.$$

Remark 4.4. Equivalently, C is Steinberg if $C \cong [\operatorname{Spec} \bar{\mathbb{F}}_p[s]/\mathbb{G}_m]$ and C is non-Steinberg if $C \cong [\operatorname{Spec} \bar{\mathbb{F}}_p[t^{\pm 1}]/\mathbb{G}_m]$.

Proposition 4.5. $\mathcal{W}_B$ is equidimensional of dimension 0, and there is a canonical morphism $\mathcal{X}_B \to \mathcal{W}_B$. For each irreducible component $C \subset \mathcal{W}_B$, $C \times_{\mathcal{W}_B} X_B \to C$ is a genuine $\mathbb{G}_a^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$-torsor.

Proof. Note that $\mathcal{X}_{\mathbb{G}_m}$ is a disjoint union of $[\mathbb{G}_m/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts trivially on $\mathbb{G}_m$, indexed by Serre weights for GL_1. Irreducible components of $X_{\mathbb{G}_m}$ are isomorphic to $\mathbb{G}_m$ and the universal family over each $\mathbb{G}_m$ is the family of unramified twists of a Galois character $\mathrm{Gal}_F \to \bar{\mathbb{F}}_p^\times$.

As a consequence, W_B is equidimensional of dimension 1 by Lemma 2.5. So, $\mathcal{W}_B = [W_B/\mathbb{G}_m]$ is equidimensional of dimension 0. The morphism $X_B \to W_B$ is defined and shown to be surjective by Lemma 2.2, and clearly descends to $\mathcal{X}_B \to \mathcal{W}_B$.

Let $C \subset \mathcal{W}_B$ be an irreducible component. By Lemma 2.8 and Lemma 2.2, $X_B(C) \to C$ is surjective and is a pseudo $\mathbb{G}_a^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$-torsor: we can define a group action $\mathbb{G}_a^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} \times_C X_B(C) \cong X_B(C) \times_C X_B(C)$. In general, a pseudo $\mathbb{G}_a$-torsor does not have to be a genuine $\mathbb{G}_a$-torsor. If we can construct a section $C \to X_B(C)$ fpqc locally, then $X_B(C) \to C$ is a genuine $\mathbb{G}_a^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$-torsor. If $C \subset \mathcal{W}_B$ is not a Steinberg component, then there exists the trivial section $C \to X_B(C)$ that corresponds to the zero extension classes $\begin{bmatrix} * & 0 \\ & * \end{bmatrix}$.

It remains to consider the Steinberg component $[\operatorname{Spec} \bar{\mathbb{F}}_p[s]/\mathbb{G}_m] \cong C_{\mathrm{st}} \subset \mathcal{W}_B$. We complete the proof by explicitly constructing a local section in flat topology. C_{st} maps to the cyclotomic character point in $\mathcal{X}_{\mathbb{G}_m}$; write M_{cyc} for the étale (φ, Γ)-module that corresponds to the mod p cyclotomic Galois character. Let $(c_f, c_g) \in Z^1(M_{\mathrm{cyc}}, \bar{\mathbb{F}}_p)$ be a trés ramifié cocycle. So, we do get a section $\operatorname{Spec} \bar{\mathbb{F}}_p[s] \to X_B(C_{\mathrm{st}})$ corresponding to the 1-cocycle $(s\ c_f, s\ c_g) \in Z^1(M_{\mathrm{cyc}} \otimes_{\bar{\mathbb{F}}_p} \bar{\mathbb{F}}_p[s], \bar{\mathbb{F}}_p[s])$. We don't know if the composition $[\operatorname{Spec} \bar{\mathbb{F}}_p[s]/\mathbb{G}_m] \to X_B(C_{\mathrm{st}}) \to C_{\mathrm{st}} \cong [\operatorname{Spec} \bar{\mathbb{F}}_p[s]/\mathbb{G}_m]$ is an isomorphism. Nevertheless, it is clear that it induces a surjective group scheme homomorphism $\operatorname{Spec} \bar{\mathbb{F}}_p[s] \to \operatorname{Spec} \bar{\mathbb{F}}_p[s]$. Group scheme homomorphisms $\operatorname{Spec} \bar{\mathbb{F}}_p[s] \to \operatorname{Spec} \bar{\mathbb{F}}_p[s]$ are classified by polynomials $s \mapsto \sum_{i=0}^N a_i s^{p^i}$ for $a_0, \ldots, a_N \in \bar{\mathbb{F}}_p$, and they are all finite flat morphisms (except for the zero morphism). Thus $X_B(C_{\mathrm{st}}) \to C_{\mathrm{st}}$ is indeed an fppf $\mathbb{G}_a^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$ -torsor, and we are done. □

Theorem 4.6. We have

$$X_B \cong W_B \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]},$$

and thus the morphism

$$\mathrm{WD} : \mathcal{X}_B \to \mathcal{W}_B$$

is smooth of relative dimension $[F : \mathbb{Q}_p]$, and induces a bijection of irreducible components.

Proof. Note that all irreducible components of $\mathcal{W}_B$ that do not intersect with the Steinberg component are connected components. So, by Proposition 4.5, it remains to restrict to the connected component containing the Steinberg component, which is isomorphic to $[\operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1},s]/(s(t-1))/\mathbb{G}_m]$ and is the union of two irreducible components $[\operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}]/\mathbb{G}_m]$ and $[\operatorname{Spec}\bar{\mathbb{F}}_p[s]/\mathbb{G}_m]$. The morphism

$$X_B \times_{W_B} \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}] \to \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}]$$

is relatively representable by a vector bundle of rank $([F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p])$, and is thus a relatively representable by a trivial vector bundle by Swan's theorem (that any finite projective module over $\mathbb{G}_m$ is finite free). So,

$$X_B \times_{W_B} \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}] \cong \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}] \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} =: \mathcal{Y},$$

which induces a natural morphism

$$\mathcal{Y} \times_{\operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}]} \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1},s]/(s(t-1)) \to X_B \times_{W_B} \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1},s]/(s(t-1))$$

which is an isomorphism by Lemma 4.8 below. $\square$

Indeed, we have explicitly written down coordinate rings for X_B. For each irreducible component $C = [\operatorname{Spec}\mathcal{O}_C/\mathbb{G}_m] \subset W_B$, Swan's theorem implies $X_B(C) \cong \operatorname{Spec}\mathcal{O}_C \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$. It remains to understand how the Steinberg component $X_B(C_{\mathrm{st}}) \cong \operatorname{Spec}\bar{\mathbb{F}}_p[s] \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$ is glued to a non-Steinberg component $X_B(C') \cong \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1}] \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$. From the proof of Theorem 4.6, the glued scheme is simply given by

$$X_B(C_{\mathrm{st}} \cup C') \cong \operatorname{Spec}\bar{\mathbb{F}}_p[t,t^{-1},s]/(s(t-1)) \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}.$$

4.2. **The universal family over $X_B(C_{\mathrm{st}}\cup C')$.** Next, we write down the universal family for $X_B(C_{\mathrm{st}}\cup C')$ in explicit cocycle systems.

Choose a 2-cochain $b_{\mathrm{std}} \in C^2_{\mathrm{Herr}}(U_{\gamma^\vee}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]/(t-1)}))$ such that

$$[b_{\mathrm{std}}]_{t=1} = 1 \in H^2(\mathrm{Gal}_F, \bar{\mathbb{F}}_p(1)) \cong \bar{\mathbb{F}}_p.$$

Using the embedding

$$\bar{\mathbb{F}}_p[t^{\pm 1}]/(t-1) = \bar{\mathbb{F}}_p \hookrightarrow \bar{\mathbb{F}}_p[t^{\pm 1}],$$

we treat b_{std} as a cocycle in $C^2_{\mathrm{Herr}}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]})$. Note that b_{std} is a 2-coboundary away from the cyclotomic point $(t-1)$:

$$b_{\mathrm{std}} = \frac{1}{(t-1)^{n_0}} d^1_{\mathrm{Herr}}(c_{\mathrm{std}}) \tag{5}$$

for some $c_{\mathrm{std}} \in C^1_{\mathrm{Herr}}(U_{\gamma^\vee}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]}))$.

Lemma 4.7. *If n_0 is the smallest possible integer, then $[c_{\mathrm{std}}]_{t=1} \neq 0 \in H^1(\mathrm{Gal}_F, \bar{\mathbb{F}}_p(1))$.*

Proof. If $[c_{\mathrm{std}}]_{t=1} = 0$, then c_{std} is a 1-coboundary at $t=1$. After conjugation by an element of $U(\bar{\mathbb{F}}_p)$, we can assume c_{std} vanishes (as a cochain) at $t=1$. Since $\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]}/(t-1) = \mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]/(t-1)}$, we conclude that $c_{\mathrm{std}} = (t-1)c'$ for some $c' \in C^1_{\mathrm{Herr}}$, contradicting the minimality of n_0. $\square$

From the isomorphism $X_B(C') \cong C' \times \mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$, we obtain global sections

$$\{c_1, \ldots, c_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}\} \subset Z^1_{\mathrm{Herr},+}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t,t^{-1}]})$$

generating $H^1_{\mathrm{Herr},+}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t,t^{-1}]})$. We thus consider the universal cochain

$$c_{\mathrm{univ}} := x_1c_1 + x_2c_2 + \cdots + x_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}c_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} + sc_{\mathrm{std}},$$

which is a cocycle if and only if $s(t-1)^{n_0}=0$. We have thus defined a morphism

$$\operatorname{Spec}\bar{\mathbb{F}}_p[t^{\pm 1}, x_1, \dots, x_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}, s]/(s(t-1)^{n_0}) \to X_B(C' \cup C_{\mathrm{st}}),$$

which induces a bijection of $\bar{\mathbb{F}}_p$-points by Lemma 4.7, and is an isomorphism away from the $t=1$ locus.

Lemma 4.8. *We must have $n_0 = 1$ and $\operatorname{Spec}\bar{\mathbb{F}}_p[t^{\pm 1}, x_1, \dots, x_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}, s]/(s(t-1)^{n_0}) \cong X_B(C' \cup C_{\mathrm{st}})$.*

Proof. Put $R = \bar{\mathbb{F}}_p[t^{\pm 1}]$, $u = t-1$, and $d = [F:\mathbb{Q}_p] + [k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]$. On the connected component under consideration the perfect modified Herr complex splits, in the derived category of R-modules, as a free summand of rank d in degree one and the two-term resolution

$$[R \xrightarrow{u} R]$$

of its degree-two torsion. This follows from Lemma 2.8, the fact that $H^2 = R/(u)$, and Smith normal form over the PID R. The relative derived vector space represented by this complex therefore has classical coordinate ring

$$R[x_1, \dots, x_d, s]/(su).$$

The universal cocycle constructed above induces this identification: the free summand gives the x_i, while c_{std} is the generator dual to the resolution and gives s. Thus its differential is exactly su, not su^{n_0}; in particular $n_0 = 1$. This computes the completed local ring at the crossing and the ordinary coordinate ring simultaneously, proving the claimed isomorphism without inferring a singular local ring from tangent spaces alone. ☐

4.3. **The universal family over $X_B(C_{\mathrm{triv}})$.** Now, let C_{triv} be the (non-Steinberg) irreducible component of trivial inertial type.

We have $X_B(C_{\mathrm{triv}}) \cong \operatorname{Spec}\bar{\mathbb{F}}_p[t^{\pm 1}, x_1, \dots, x_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}]$. Let $c_{\mathrm{un}} \in Z^1_{\mathrm{Herr}}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]})$ be an unramified cocycle with $[c_{\mathrm{un}}]$ being nontrivial cohomology class in H^1_{Herr}. Then c_{un} is a coboundary away from $t=1$. So, $H^1_{\mathrm{Herr}}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]})/\bar{\mathbb{F}}_p c_{\mathrm{un}}$ is a finite free $\bar{\mathbb{F}}_p[t^{\pm 1}]$-module of rank $[F:\mathbb{Q}_p]$. We can choose global sections

$$\{c_1, \dots, c_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}\} \subset Z^1_{\mathrm{Herr},+}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t,t^{-1}]})$$

such that

- $\{[c_1], \dots, [c_{[F:\mathbb{Q}_p]}]\}$ form a basis for $H^1_{\mathrm{Herr}}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]})/\bar{\mathbb{F}}_p[c_{\mathrm{un}}]$
- $\{[c_1], \dots, [c_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}]\}$ form a basis for $H^1_{\mathrm{Herr},+}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t^{\pm 1}]})$ and that
- $c_{[F:\mathbb{Q}_p]+1}$ lifts c_{un}.

The universal family is thus of the form

$$c^{\mathrm{univ}} = x_1 c_1 + \cdots + x_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} c_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} \in Z^1_{\mathrm{Herr}}(\mathbf{E}_{F,\mathcal{O}_{C_{\mathrm{triv}}}}).$$

5. Example: PGL_3

Write $B = U \rtimes \breve{T} = B_{\mathrm{PGL}_3}$ for the Borel of PGL_3.

Let δ_1, δ_2 be the simple roots and $\gamma = \delta_1 + \delta_2$ be the highest root. So, $U = U_{\delta_1^\vee} \times U_{\delta_2^\vee} \times U_{\gamma^\vee}$ and $\breve{T} \cong \mathbb{G}_{m,1} \times \mathbb{G}_{m,2}$ where $\mathbb{G}_{m,i} \cong \mathbb{G}_m$ acts on $U_{\delta_i^\vee}$ by the weight 1 character for $i = 1, 2$. Write $\mathbb{G}_{m,\gamma}$ for the image of $\mathbb{G}_m \xrightarrow{t \mapsto (t,t)} \mathbb{G}_{m,1} \times \mathbb{G}_{m,2} \cong \breve{T}$.

Consider the morphism

$$X_B \to X_{U_{\delta_1} \rtimes \mathbb{G}_{m,1}} \times X_{U_{\delta_2} \rtimes \mathbb{G}_{m,2}} \xrightarrow{\mathrm{WD} \times \mathrm{WD}} W_{U_{\delta_1} \rtimes \mathbb{G}_{m,1}} \times W_{U_{\delta_2} \rtimes \mathbb{G}_{m,2}},$$

which we denote by $\widetilde{\mathrm{WD}}$. Let $C_i \subset \mathcal{W}_{U_{\delta_i^\vee} \rtimes \mathbb{G}_{m,i}}$ be an irreducible component, $i = 1, 2$.

Set $C = C_1 \times C_2$ and $X_B(C) := \widetilde{\mathrm{WD}}^{-1}(C)$. For ease of notation, we assume $F = \mathbb{Q}_p$.

5.1. **The case where both C_1 and C_2 are non-Steinberg.** To simplify discussion, we assume neither C_1 or C_2 intersects with the Steinberg component.

We have

$$X_{U_{\delta_i} \rtimes \mathbb{G}_{m,i}}(C_i) \cong \operatorname{Spec} \bar{\mathbb{F}}_p[t_i, t_i^{-1}][x_{\delta_i,1}, x_{\delta_i,2}].$$

The parameters x_{i1}, x_{i2} correspond to the parametric 1-cocycles

$$c_{\delta_i,1}, c_{\delta_i,2} \in Z^1_{\mathrm{Herr}}(\mathbf{E}_{F, \bar{\mathbb{F}}_p[t_i^{\pm 1}]}).$$

If $H^2_{\mathrm{Herr}}(U_{\gamma^\vee}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]})) = 0$, then we have

$$\begin{aligned} X_B(C) &\cong (X_{U_{\delta_1} \rtimes \mathbb{G}_{m,1}}(C_1) \times X_{U_{\delta_2} \rtimes \mathbb{G}_{m,2}}(C_2)) \times_{X_{\mathbb{G}_{m,\gamma}}} X_{U_\gamma \rtimes \mathbb{G}_{m,\gamma}}(C_{12}) \\ &\cong \operatorname{Spec} \bar{\mathbb{F}}_p[t_1, t_1^{-1}][x_{\delta_1,1}, x_{\delta_1,2}] \times \operatorname{Spec} \bar{\mathbb{F}}_p[t_2, t_2^{-1}][x_{\delta_2,1}, x_{\delta_2,2}] \times \mathbb{A}^2, \end{aligned}$$

where C_{12} is the image of

$$C_1 \times C_2 \hookrightarrow X_{\mathbb{G}_{m,1}} \times X_{\mathbb{G}_{m,2}} \to X_{\mathbb{G}_{m,\gamma}}.$$

So, we assume $H^2_{\mathrm{Herr}}(U_{\gamma^\vee}) \neq 0$ at $t_1 = t_2 = 1$. We write

$$b_{\mathrm{std}} \in C^2_{\mathrm{Herr}}(U_{\gamma^\vee}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]})) \qquad c_{\mathrm{std}} \in C^1_{\mathrm{Herr}}(U_{\gamma^\vee}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]}))$$

for suitable base change of the cochains defined in Eq. (5).

We fix 1-cochains $c_{\gamma,[11]}, c_{\gamma,[12]}, c_{\gamma,[21]}, c_{\gamma,[22]}$ such that

$$c_{\delta_1,i_1} \cup c_{\delta_2,i_2} - a_{i_1,i_2} b_{\mathrm{std}} = d^1_{\mathrm{Herr}}(c_{\gamma,[i_1 i_2]}) \tag{6}$$

where $a_{i_1,i_2} \in \bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]$ are unique scalars, which we can normalize to ensure $a_{*,*} \in \{0,1\} \subset \bar{\mathbb{F}}_p$:

Lemma 5.1. (1) We can normalize $c_{\delta_i,j}$ to ensure $a_{11} = 1, a_{12} = a_{21} = a_{22} = 0$.

(2) The left hand side of Eq. (6) is a coboundary.

Proof. (1) We can assume $[c_{\delta_i,1}] \neq 0$ are non-trivial Herr complex cohomology classes and $[c_{\delta_i,2}] = 0$ are trivial Herr complex cohomology classes. By local Tate duality, a_{11} is pointwise a unit over $\operatorname{Spec} \bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]$. Thus $a_{11} \in \bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]^\times$ and is of the form $t_1^{e_1} t_2^{e_2} \bar{\mathbb{F}}_p^\times$. We can renormalize $c_{\delta_i,j}$ to make $a_{11} = 1$.

(2) Since $H^2_{\mathrm{Herr}} \cong \bar{\mathbb{F}}_p[t_1^{\pm1}, t_2^{\pm1}]/(t_1 t_2 - 1) \cong \bar{\mathbb{F}}_p[t^{\pm1}]$ and B^2_{Herr} is the image of d^1_{Herr}, it suffices to show the left hand side of Eq. (6) (which we denote by c_{LHS}) is a coboundary mod $(t_1 t_2 - 1)$. We have

$$[c_{\mathrm{LHS}}] \mod (t_1 t_2 - 1) \in H^2_{\mathrm{Herr}}(\mathbf{E}_{F,\bar{\mathbb{F}}_p[t_1^{\pm1}]}) \cong \bar{\mathbb{F}}_p[t^{\pm1}]$$

is fiberwise 0 over $\operatorname{Spec} \bar{\mathbb{F}}_p[t_1^{\pm1}]$. So, we are done as $\bar{\mathbb{F}}_p[t^{\pm1}]$ is reduced. $\square$

The universal family we set

$$\begin{aligned} c_{\delta_1}^{\mathrm{univ}} &= x_{\delta_1,1}c_{\delta_1,1} + x_{\delta_1,2}c_{\delta_1,2} \\ c_{\delta_2}^{\mathrm{univ}} &= x_{\delta_2,1}c_{\delta_2,1} + x_{\delta_2,2}c_{\delta_2,2} \\ c_{\gamma}^{\mathrm{univ}} &= s_\gamma c_{\mathrm{std}} + x_{\delta_1,1}x_{\delta_2,1}c_{\gamma,[11]} + x_{\delta_1,1}x_{\delta_2,2}c_{\gamma,[12]} + x_{\delta_1,2}x_{\delta_2,1}c_{\gamma,[21]} + x_{\delta_1,2}x_{\delta_2,2}c_{\gamma,[22]}. \end{aligned}$$

where $s_\gamma, x_{\delta_1,1}, x_{\delta_1,2}, x_{\delta_2,1}, x_{\delta_2,2}$ are free parameters. Write $\chi_i^{\mathrm{univ}} \in X_{\mathbb{G}_{m,i}}(\mathcal{O}_{X_{\mathbb{G}_{m,i}}})$ for the universal rank 1 étale (φ, Γ)-modules. Then

$$\begin{bmatrix} \chi_1^{\mathrm{univ}}\chi_2^{\mathrm{univ}} & \chi_2^{\mathrm{univ}}c_{\delta_1}^{\mathrm{univ}} & c_\gamma^{\mathrm{univ}} \\ & \chi_2^{\mathrm{univ}} & c_{\delta_2}^{\mathrm{univ}} \\ & & 1 \end{bmatrix}$$

is a valid étale (φ, Γ)-module if and only if

$$s_\gamma(t_1t_2 - 1) + x_{\delta_1,1}x_{\delta_2,1} = 0.$$

Lemma 5.2. *We have*

$$\operatorname{Spec} \bar{\mathbb{F}}_p[t_1^{\pm 1}, t_2^{\pm 1}, x_{\delta_1,1}, x_{\delta_1,2}, x_{\delta_2,1}, x_{\delta_2,2}, s_\gamma]/(s_\gamma(t_1t_2 - 1) + x_{\delta_1,1}x_{\delta_2,1}) \times \mathbb{A}^2 \xrightarrow{\cong} X_B(C)$$

Proof. The morphism is defined through the explicit universal family, and is clearly an isomorphism. □

5.2. **The case where C_1 is Steinberg and C_2 is non-Steinberg.** We omit the completely similar details. The answer is

$$X_B(C) \cong \operatorname{Spec} \bar{\mathbb{F}}_p[s, t_2^{\pm 1}, x_{\delta_1,1}, x_{\delta_1,2}, x_{\delta_2,1}, x_{\delta_2,2}, s_\gamma]/(s_\gamma(t_2 - 1) + x_{\delta_1,1}x_{\delta_2,1} + x_{\delta_1,2}x_{\delta_2,2}) \times \mathbb{A}^2.$$

We also leave it as an exercise to the reader how the explicit presentations of various irreducible components $X_B(C)$ are glued to each other.

6. Weil-Deligne stacks and gauge cochains

We return to the setup of general reductive groups and keep the notation of Section 3.

6.1. **Weil-Deligne stacks.** Consider the morphism

$$\prod_{\delta \in \Delta} \delta : \check{T} \to \prod_{\delta \in \Delta} \mathbb{G}_{m,\delta}$$

where $\mathbb{G}_m \cong \mathbb{G}_{m,\delta}$ is the multiplicative group that acts trivially on $U_{\delta'^\vee}$ for $\delta \neq \delta' \in \Delta$ and acts by the root δ on $U_{\delta^\vee}$.

We have the following change of group morphism

$$\mathcal{X}_{\bar{U}_1 \rtimes \check{T}} \to \prod_{\delta \in \Delta} \mathcal{X}_{U_{\delta^\vee} \rtimes \mathbb{G}_{m,\delta}},$$

which we also denote by $\prod_{\alpha \in \Delta} \alpha$ by abuse of notation. By Definition 4.2, we have a canonical Weil-Deligne stack $\mathcal{W}_{U_{\delta^\vee} \rtimes \mathbb{G}_{m,\delta}}$ fitting in the diagram:

$$\begin{tikzcd} \mathcal{X}_{U_{\delta^\vee} \rtimes \mathbb{G}_{m,\delta}}, \arrow[rr] \arrow[dr] & & \mathcal{W}_{U_{\delta^\vee} \rtimes \mathbb{G}_{m,\delta}}, \arrow[dl] \\ & \mathcal{X}_{\mathbb{G}_{m,\delta}} & \end{tikzcd}$$

Definition 6.1. Define
$$\mathcal{W}_{\breve{B}} := \mathcal{X}_{\breve{T}} \underset{\prod_{\delta\in\Delta}\mathcal{X}_{\mathbb{G}_{m,\delta}}}{\times} \prod_{\delta\in\Delta} \mathcal{W}_{U_{\delta^\vee}\rtimes\mathbb{G}_{m,\delta}}$$

Lemma 6.2. *$\mathcal{W}_{\breve{B}}$ is an algebraic stack of finite type over $\bar{\mathbb{F}}_p$, and is equidimensional of dimension 0. Moreover, there is canonical morphism*
$$\mathrm{tr}_1\,\mathrm{WD} : \mathrm{tr}_1\,\mathcal{X}_{\breve{B}} \to \mathcal{W}_{\breve{B}}$$
whose fibers of irreducible algebraic stacks of constant dimension $[F:\mathbb{Q}_p]\#\Delta$. In particular, $\mathrm{tr}_1\,\mathrm{WD}$ induces a bijection of irreducible components.

Proof. Note that $\mathcal{X}_{\breve{T}} \cong \mathcal{X}_{\mathbb{G}_m}^{\times \dim \breve{T}}$ is equidimensional of dimension 0, and that fibers of $\mathcal{X}_{\breve{T}} \to \prod_{\alpha\in\Delta}\mathcal{X}_{\mathbb{G}_{m,\delta}}$ are irreducible of constant dimension 0. The lemma now follows from Proposition 4.5. □

6.2. **Gauge cochains.** Let C_δ be an irreducible component of the destackification $W_{U_{\delta^\vee}\rtimes\mathbb{G}_{m,\delta}}$ for each $\delta\in\Delta$. Write
$$C_\delta = \begin{cases} \mathrm{Spec}\,\bar{\mathbb{F}}_p[t_\delta^{\pm1}] & \text{non-Steinberg} \\ \mathrm{Spec}\,\bar{\mathbb{F}}_p[s_\delta] & \text{Steinberg} \end{cases}.$$
Write $\mathrm{Spec}\,R := \mathrm{Spec}\,\bar{\mathbb{F}}_p[t_\delta^{\pm1} : C_\delta \text{ non-Steinberg}]$. If $\alpha = \sum_{\delta\in\Delta} m_\delta\delta$ for $m_\delta\in\mathbb{Z}$, then set $t_\alpha := \prod t_\delta^{m_\delta}$ and $\mathbb{G}_{m,\alpha} := \mathrm{Spec}\,\bar{\mathbb{F}}_p[t_\alpha^{\pm1}]$.

For each $\alpha\in\Phi^+$, let C'_α be the unique non-Steinberg component of $W_{U_{\alpha^\vee}\rtimes\mathbb{G}_{m,\alpha}}$ that intersects with the image of $\prod_{\delta\in\Delta} C_\delta$. We fix an isomorphism
$$X_{U_{\alpha^\vee}\rtimes\mathbb{G}_{m,\alpha}}(C'_\alpha) \cong C'_\alpha \times \mathrm{Spec}\,\bar{\mathbb{F}}_p[x_{\alpha,1},\ldots,x_{\alpha,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}],$$
where the free parameter $x_{\alpha,i}$ corresponds to a fixed parametric cocycle
$$c_{\alpha,i} \in Z^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F,R})).$$

We make the convention that
$$[c_{\alpha,[F:\mathbb{Q}_p]+1}] = 0 \in H^1_{\mathrm{Herr}}(U_{\alpha^\vee})$$
unless C_α has trivial inertial type, in which case we arrange it so that
$$[c_{\alpha,[F:\mathbb{Q}_p]+1}] = 0 \in H^1_{\mathrm{Herr}}(U_{\alpha^\vee}) \tag{7}$$
away from the locus $t_\alpha = 1$ and is a nontrivial unramified class at $t_\alpha = 1$. We also assume $[c_{\alpha,[F:\mathbb{Q}_p]+j}] = 0 \in H^1_{\mathrm{Herr}}(U_{\alpha^\vee})$ for $j>1$.

If $\alpha = \gamma+\beta$ and C_α intersects with the Steinberg component, then the matrix
$$J = J_{\beta,\gamma} = ([c_{\beta,i}]\cup[c_{\gamma,j}])_{i,j\leqslant[F:\mathbb{Q}_p]} \in \mathrm{Mat}_{[F:\mathbb{Q}_p]\times[F:\mathbb{Q}_p]}(R/(t_\alpha-1))$$
which is pointwise a unit. So, $\det(J)\in t_\beta^{\mathbb{Z}}\bar{\mathbb{F}}_p^\times$.

Remark 6.3 (Torus coordinates)**.** For two roots α and β, consider the tautological isomorphism
$$\bar{\mathbb{F}}_p[t_\alpha^{\pm1}] \longrightarrow \bar{\mathbb{F}}_p[t_\beta^{\pm1}], \qquad t_\alpha \longmapsto t_\beta.$$
We group the roots according to whether this identifies their universal rank-1 modules, and pair two groups when their universal modules are dual under local Tate duality.

The coordinates are normalized so that $t_\alpha = 1$ is the trivial or cyclotomic point. If two universal modules are dual, then their specializations at $t_\alpha = t_\beta = 1$ are also dual.

Lemma 6.4 (Normalization of cup products)**.** *The cocycles may be chosen so that*

$$J_{\beta,\gamma} = I_{[F:\mathbb{Q}_p]} \in \mathrm{Mat}_{[F:\mathbb{Q}_p]\times[F:\mathbb{Q}_p]}(\bar{\mathbb{F}}_p).$$

Proof. If the two groups are distinct, choose bases independently so that the perfect pairing has the prescribed matrix, and transport these choices through the tautological isomorphisms. The resulting pairing matrices have entries in $\bar{\mathbb{F}}_p$.

In the self-dual case, the pairing is alternating and nondegenerate. We can thus normalize it so that $(c_{\beta,i} \cup c_{\beta,j})$ is a aniti-diagonal anti-symmetric constant matrix.

We apply coloring to the set of roots whose universal modules are tautologically identified with that of β. Color β by white; color γ by black (or white) if $\alpha = \beta' + \gamma$ satisfies $H^2_{\mathrm{Herr}}(U_{\alpha^\vee}) \neq 0$ and that β' is white (or black).

No root can be both white or black, as it violates the alternating property of local Tate duality. We transport to the white roots the chosen cocycles and transport to the black roots the dual cocycles. $\square$

We denote by $c_{\alpha,\mathrm{std}}$ and $b_{\alpha,\mathrm{std}}$ the cochains in Eq. (5) after substitution $t \mapsto t_\alpha$.

Definition 6.5. We recursively define finitely many 1-cochains $\mathfrak{C}_{\Xi_{\alpha,k}} = \{c_{\alpha,\xi}\} \subset C^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F,R}))$, $\xi \in \Xi_{\alpha,k}$ for $k = 1, \ldots, h_{\check{G}}$.

We set

$$\Xi_{\alpha,1} = \begin{cases} \{1, 2, \ldots, [F:\mathbb{Q}_p] + [k_{F_{\mathrm{cyc}}} : \mathbb{F}_p], \mathrm{std}\} & C_\alpha \text{ intersects with the Steinberg component} \\ \{1, 2, \ldots, [F:\mathbb{Q}_p] + [k_{F_{\mathrm{cyc}}} : \mathbb{F}_p]\} & \text{otherwise} \end{cases}$$

and $\mathfrak{C}_{\Xi_{\alpha,1}} = \{c_{\alpha,\xi} | \xi \in \Xi_{\alpha,1}\}$.

Suppose $\mathfrak{C}_{\Xi_{\alpha,k}}$ is already defined. For each sequence of roots $\beta_1 + \cdots + \beta_s = \alpha$ and elements $c_{\beta_i,\xi_i} \in \mathfrak{C}_{\Xi_{\beta_i,k}}$, we choose an element $c_{\alpha,[\xi_1,\xi_2,\ldots,\xi_s]} \in C^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F,R}))$ such that

$$[c_{\beta_1,\xi_1}, \ldots, c_{\beta_s,\xi_s}] = \begin{cases} a_{\alpha,[\xi_1,\ldots,\xi_s]} b_{\alpha,\mathrm{std}} - d^1_{\mathrm{Herr}}(c_{\alpha,[\xi_1,\xi_2,\ldots,\xi_s]}) & C_\alpha \text{ intersects with the Steinberg component} \\ -d^1_{\mathrm{Herr}}(c_{\alpha,[\xi_1,\xi_2,\ldots,\xi_s]}) & \text{otherwise} \end{cases}$$

where $a_{\alpha,*} \in R$ is the unique scalar that makes the equation hold, and $[*, \ldots, *]$ is the higher cup product defined in Definition 3.11. We set

$$\Xi_{\alpha,k+1} = \Xi_{\alpha,k} \cup \{[\xi_1, \ldots, \xi_k], \xi_i \in \mathfrak{C}_{\Xi_{\beta_i,k}}\}$$

and

$$\mathfrak{C}_{\Xi_{\alpha,k+1}} = \{c_{\alpha,\xi} : \xi \in \Xi_{\alpha,k+1}\}.$$

We call elements of $\mathfrak{C}_{\Xi_{*,*}}$ the *gauge cochains.*

6.3. **Universal families.** We recursively define universal families over

$$X_{\leqslant\alpha}(C) := X_{\leqslant\alpha} \times_{X_B} X_B(C),$$

where $C = (C_\delta : \delta \in \Delta)$ is the fixed irreducible component of W_B.

Suppose

$$c^{\mathrm{univ}}_\beta \in C^1_{\mathrm{Herr},+}(U_{\beta^\vee}(\mathbf{E}_{F,\mathcal{O}_{X_{<\alpha}(C)}}))$$

are the universal coordinates such that $\exp_{\leqslant h_{\check{G}}}(\sum_{\beta<\alpha} c_\beta)$ corresponds to the universal family for $X_{<\alpha}(C)$ using the notation of Theorem 3.9. Let $\mu_{\beta,\xi} \in \mathcal{O}_{X_{<\alpha}(C)}$ denote the coefficient $c_{\beta,\xi}$ in c^{univ}_β.

Recall that the higher cup product $[*,\dots,*]$ is defined as in Definition 3.11. We set

$$c_\alpha^{\mathrm{univ}} =\Big(\sum_{\beta_1+\dots+\beta_m=\alpha}\frac{1}{m!}\{c_{\beta_1}^{\mathrm{univ}},\dots,c_{\beta_m}^{\mathrm{univ}}\}\Big)+\sum_{i=1}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} x_{\alpha,i}c_{\alpha,i}$$
$$+\begin{cases} s_\alpha c_{\alpha,\mathrm{std}} & C_\alpha \text{ intersects with the Steinberg component}\\ 0 & \text{otherwise}\end{cases}$$

where $\{s_\alpha, x_{\alpha,1},\dots,x_{\alpha,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}\}$ are the new free parameters and $\{*,\dots,*\}$ is the multilinear map that expands

$$\{c_{\beta_1,\xi_1},\dots,c_{\beta_m,\xi_m}\}:=c_{\alpha,[\xi_1,\dots,\xi_m]}.$$

Lemma 6.6. If C_α intersects with the Steinberg component, then $\exp_{\leqslant h_{\breve{G}}}(\sum_{\beta<\alpha}c_\beta)$ corresponds to the universal family for $X_{<\alpha}(C)$ (in the notation of Theorem 3.9) if and only if

$$r_\alpha := s_\alpha(t_\alpha-1)+\sum_{\beta_1+\dots+\beta_m=\alpha,\xi_1,\dots,\xi_m}\frac{1}{m!}a_{\alpha,[\xi_1,\dots,\xi_m]}\prod_{j=1}^m \mu_{\beta_j,\xi_j}=0.$$

Proof. Combine Theorem 3.9 and Lemma 4.7. □

Corollary 6.7. There is an isomorphism

$$X_{\leqslant\alpha}(C)\cong\begin{cases} X_{<\alpha}(C)[s_\alpha]/(r_\alpha)\times\mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} & C_\alpha \text{ intersects with the Steinberg component}\\ X_{<\alpha}(C)\times\mathbb{A}^{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} & \text{otherwise.}\end{cases}$$

Proof. Follows immediately from Lemma 6.6. □

Corollary 6.8. If all irreducible components of $X_{<\alpha}(C)$ have dimension $\geqslant d$, then all irreducible components of $X_{\leqslant\alpha}(C)$ have dimension $\geqslant d+[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]$.

Proof. It follows from Krull's Hauptidealsatz (cf. [Stacks, Tag 0EPZ]). □

Corollary 6.9. All irreducible components of $\mathcal{X}_{\breve{B}}$ have dimension $\geqslant [F:\mathbb{Q}_p]\#\Phi^+$.

If there is a stratification of $\mathcal{X}_{\breve{B}}$ by locally closed substacks such that each stratum has dimension $\leqslant [F:\mathbb{Q}_p]\#\Phi^+$, then the irreducible components of $\mathcal{X}_{\breve{B}}$ are precisely the closure of the top-dimensional strata.

Proof. Argue inductively using Corollary 6.8. □

7. Fine strata and cone models

7.1. **Minimal normal subgroups.**

Lemma 7.1. *Let $\Gamma\subset\breve{B}(\bar{\mathbb{F}}_p)$ be a subgroup. Let $K_1,K_2\subset U$ be $\breve{T}$-stable normal subgroups. If there exists an element $b_i\in\breve{B}(\bar{\mathbb{F}}_p)$ ($i=1,2$) such that $b_i\Gamma b_i^{-1}\subset(K_i\rtimes\breve{T})(\bar{\mathbb{F}}_p)$. Then there exists $b\in\breve{B}(\bar{\mathbb{F}}_p)$ such that $b\Gamma b^{-1}\subset(K_1\cap K_2)\rtimes\breve{T}(\bar{\mathbb{F}}_p)$.*

Proof. We argue by induction on the descending central series of U.

Recall the filtration $U_k\subset U$, where $\operatorname{Lie}U_1=\operatorname{Lie}U$ and $\operatorname{Lie}U_k=[\operatorname{Lie}U,\operatorname{Lie}U_{k-1}]$ for $k>1$. Let

$$\bar{U}_k:=\frac{U_k}{(K_1\cap K_2\cap U_k)U_{k+1}}.$$

We prove by induction on k that there exists $b_k \in \breve{B}(\mathbb{F}_p)$ such that $b_k \Gamma b_k^{-1} \subset ((K_1 \cap K_2)U_k) \rtimes \breve{T}(\mathbb{F}_p)$. For $k = 1$, this is trivial since $U_1 = U$.

Assume this holds for some k. Replacing Γ by $b_k \Gamma b_k^{-1}$, we may assume $\Gamma \subset ((K_1 \cap K_2)U_k) \rtimes \breve{T}(\mathbb{F}_p)$.

Pass to the quotient $\breve{B}_k := \breve{B}/(K_1 \cap K_2)U_{k+1}$. The image of Γ in $\breve{B}_k$ takes the form $\gamma = (c(\gamma), t(\gamma)) \in \bar{U}_k \rtimes \breve{T}$, where $c : \Gamma \to \bar{U}_k$.

Write the image of b_i in $\breve{B}_k$ as $b_i = u_i t_i \in \breve{B}_k(\mathbb{F}_p)$, with $u_i \in \bar{U}_k(\mathbb{F}_p)$ and $t_i \in \breve{T}(\mathbb{F}_p)$. Since $K_i \rtimes \breve{T}$ is $\breve{T}$-stable, replacing b_i by u_i preserves the hypothesis, so we may assume $b_i = u_i \in \bar{U}_k(\mathbb{F}_p)$.

Because $\bar{U}_k$ is central in $U/(K_1 \cap K_2)U_{k+1}$, conjugation by an element $u \in \bar{U}_k(\mathbb{F}_p)$ acts by:

$$(u, 1)(c, t)(u^{-1}, 1) = (c + (1 - t)u, t).$$

Define the index sets of missing roots for each subgroup at height k:

$$\Psi_i := \{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = k,\ \alpha \notin \Phi_{K_i}\}.$$

The quotient decomposes as $\bar{U}_k = \bigoplus_{\alpha \in \Psi_1 \cup \Psi_2} U_{\alpha^\vee}$. Expanding $c = \sum c_\alpha$ and $u_i = \sum u_{i,\alpha}$, the conjugation action changes coordinates independently via $c_\alpha \mapsto c_\alpha + (1 - \alpha(t))u_\alpha$.

Because conjugating γ by u_i maps it into $K_i \rtimes \breve{T}$, the resulting α-coordinates must vanish for all roots not in K_i. Therefore:

$$c_\alpha(\gamma) + (1 - \alpha(t(\gamma)))u_{i,\alpha} = 0 \quad \text{for all } \alpha \in \Psi_i.$$

We define our conjugating element $u \in \bar{U}_k(\mathbb{F}_p)$ explicitly:

$$u := \sum_{\alpha \in \Psi_2} u_{2,\alpha} + \sum_{\alpha \in \Psi_1 \backslash \Psi_2} u_{1,\alpha}.$$

Conjugating γ by this u zeroes out the α-coordinates across all of $\Psi_1 \cup \Psi_2$:

- For $\alpha \in \Psi_2$, the new coordinate is $c_\alpha(\gamma) + (1 - \alpha(t(\gamma)))u_{2,\alpha} = 0$.
- For $\alpha \in \Psi_1 \backslash \Psi_2$, the new coordinate is $c_\alpha(\gamma) + (1 - \alpha(t(\gamma)))u_{1,\alpha} = 0$.

Since all coordinates in the support of $\bar{U}_k$ vanish, $u\Gamma u^{-1}$ lies in the trivial subgroup of $\bar{U}_k$, meaning it is contained in $((K_1 \cap K_2)U_{k+1}) \rtimes \breve{T}(\mathbb{F}_p)$. This completes the induction step. □

In particular, for each Galois representation, $\bar{\rho} : \mathrm{Gal}_F \to \breve{B}(\bar{\mathbb{F}}_p)$, it makes sense to define *the minimal $\breve{T}$-stable normal subgroup* $K \subset U$ such that a $\breve{B}$-conjugate of $\bar{\rho}$ factors through $K \rtimes \breve{T}$.

Lemma 7.2. Suppose K is the minimal normal subgroup of U such that $\bar{\rho} : \mathrm{Gal}_F \to \breve{B}(\bar{\mathbb{F}}_p)$ factors through $K \rtimes \breve{T}$, up to $\breve{B}$-conjugate.

Write the composite $\mathrm{Gal}_F \to (K \rtimes \breve{T})(\bar{\mathbb{F}}_p) \to (\bar{K}_1 \rtimes \breve{T})(\bar{\mathbb{F}}_p)$ as $c\bar{\rho}^{\mathrm{ss}}$ where $\bar{\rho}^{\mathrm{ss}} : \mathrm{Gal}_F \to \breve{T}(\bar{\mathbb{F}}_p)$ is the semisimplification, and that $c = \sum_{\mathrm{ht}_K(\alpha)=1} c_\alpha \in Z^1(\mathrm{Gal}_F, \bar{K}_1(\bar{\mathbb{F}}_p))$ where each $[c_\alpha] \in H^1(\mathrm{Gal}_F, U_{\alpha^\vee}(\bar{\mathbb{F}}_p))$. We must have $[c_\alpha] \neq 0$ for each α.

Proof. If c_α is a coboundary, then we can conjugate $\bar{\rho}$ such that $\bar{\rho}$ factors through $\prod_{\beta \neq \alpha, \beta \in \Phi_K} U_{\beta^\vee} \rtimes \breve{T}$, which contradicts the minimality of K. □

7.2. **The fine stratification.** By Corollary 6.9, we don't need to compute the full $\mathcal{X}_{\breve{B}}$ — the computation of a suitable stratification suffices.

Definition 7.3. A fine configuration is a tuple (K, Ψ_0, Ψ_2) where K is a $\breve{T}$-stable normal subgroup of U and $\Psi_0, \Psi_2 \subset \Phi^+$ are subsets satisfying

- $\alpha,\beta\in\Psi_2, \alpha-\beta\in\Phi^+ \Rightarrow \alpha-\beta\in\Psi_0$, and
- $\alpha\in\Psi_2, \beta\in\Psi_0, \alpha+\beta\in\Phi^+ \Rightarrow \alpha+\beta\in\Psi_2$.

Define

$$\mathcal{D}(\Psi_0,\Psi_2)=\left\{\sum_i n_i \;\middle|\; \begin{array}{l} n_i\in\mathbb{Z}_{\geqslant 0},\quad \sum_i n_i>0, \\ \alpha=\sum_i n_i\alpha_i \text{ for some } \alpha\in\Psi_0 \text{ and } \alpha_i\in\Psi_2 \end{array}\right\}.$$

The minimal degree of (Ψ_0,Ψ_2) is the extended positive integer

$$\deg(\Psi_0,\Psi_2)=\begin{cases}\min\mathcal{D}(\Psi_0,\Psi_2), & \mathcal{D}(\Psi_0,\Psi_2)\neq\varnothing,\\ +\infty, & \mathcal{D}(\Psi_0,\Psi_2)=\varnothing.\end{cases}$$

Lemma 7.4. *Let $\bar\rho^{\mathrm{ss}}:\mathrm{Gal}_F\to\check{T}(\bar{\mathbb{F}}_p)$ be a Galois representation such that*

$$H^2(\mathrm{Gal}_F,U_{\alpha^\vee}(\bar{\mathbb{F}}_p))\neq 0 \iff \alpha\in\Psi_2,$$
$$H^0(\mathrm{Gal}_F,U_{\alpha^\vee}(\bar{\mathbb{F}}_p))\neq 0 \iff \alpha\in\Psi_0.$$

Put $\delta=\deg(\Psi_0,\Psi_2)$. If $\delta<+\infty$, then

$$[F(\mu_p):F]\mid\delta$$

and

$$[F:\mathbb{Q}_p]\geqslant\frac{p-1}{\gcd(p-1,\delta)}\geqslant\frac{p-1}{\delta}.$$

Equivalently, $[F:\mathbb{Q}_p]\delta\geqslant p-1$. If $\delta=+\infty$, the hypotheses impose no field-degree bound through the minimal degree.

Proof. Let $\bar\epsilon:\mathrm{Gal}_F\to\bar{\mathbb{F}}_p^\times$ be the mod p cyclotomic character and put

$$m=\#\bar\epsilon(\mathrm{Gal}_F)=[F(\mu_p):F].$$

By local Tate duality, the character on $U_{\alpha_i^\vee}(\bar{\mathbb{F}}_p)$ is $\bar\epsilon$ for every $\alpha_i\in\Psi_2$, whereas the character on $U_{\alpha^\vee}(\bar{\mathbb{F}}_p)$ is trivial for every $\alpha\in\Psi_0$.

Suppose that

$$\alpha=\sum_i n_i\alpha_i,\qquad \alpha\in\Psi_0,\quad \alpha_i\in\Psi_2.$$

Evaluating $\bar\rho^{\mathrm{ss}}$ on this identity gives

$$1=\bar\epsilon^{\sum_i n_i}.$$

Thus $m\mid\sum_i n_i$. Taking a relation realizing the minimum gives $m\mid\delta$. Since also $m\mid p-1$, we have $m\leqslant\gcd(p-1,\delta)$.

Let $E=F\cap\mathbb{Q}_p(\mu_p)$. Then $[E:\mathbb{Q}_p]=\frac{p-1}{m}$. As $E\subset F$, it follows that

$$[F:\mathbb{Q}_p]=[F:\mathbb{Q}_p]\geqslant\frac{p-1}{m}\geqslant\frac{p-1}{\gcd(p-1,\delta)}.$$

The final inequality follows from $\gcd(p-1,\delta)\leqslant\delta$. □

Definition 7.5. Denote by

$$X_{K\rtimes\check{T}}\langle\Psi\rangle\subset X_{K\rtimes\check{T}}$$

for the locally closed substack whose $\bar{\mathbb{F}}_p$-points are described in Lemma 7.4.

Theorem 7.6. (1) We have

$$X_{K\rtimes\breve{T}}\langle\Psi\rangle = \operatorname{Spec}\bar{\mathbb{F}}_p\begin{bmatrix} x_{\alpha,1},\dots,x_{\alpha,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}, s_\alpha & : & \alpha\in\Phi_K \\ t_\delta^{\pm 1} & : & \delta\in\Delta \end{bmatrix}/I$$

where I is generated by

$$\begin{Bmatrix} s_\alpha & : & \alpha\notin\Psi_2 \\ r_\alpha & : & \alpha\in\Psi_2, \mathrm{ht}_K(\alpha)>1 \\ t_\alpha - 1 & : & \alpha\in\Psi_0\cup\Psi_2 \end{Bmatrix}.$$

Here, r_α is defined in Lemma 6.6.

(2) Write

$$\vec{x}_\beta := (x_{\beta,1},\dots,x_{\beta,[F:\mathbb{Q}_p]+1}, s_\beta)$$
$$\vec{x}_\beta\cdot\vec{x}_\gamma := \sum_{i=1}^{[F:\mathbb{Q}_p]} x_{\beta,i}x_{\gamma,i} + x_{\beta,[F:\mathbb{Q}_p]+1}s_\gamma + x_{\gamma,[F:\mathbb{Q}_p]+1}s_\beta.$$

We can write r_α as

$$r_\alpha = s_\alpha(t_\alpha - 1) + \sum_{\beta\in\Phi_K,\delta\in\Delta_K,\delta+\beta=\alpha}\frac{1}{2}N_{\beta,\delta}\vec{x}_\beta\cdot\vec{x}_\delta + Q_\alpha$$

where Q_α has monomials divisible by $x_{\gamma_1,i_1}x_{\gamma_2,i_2}$ for $1<\mathrm{ht}(\gamma_i)<\mathrm{ht}_K(\alpha)-1$, and Q_α does not depend on $x_{\beta,*}, s_\beta$ for $\mathrm{ht}_K(\beta)=\mathrm{ht}_K(\alpha)-1$.

Proof. (1) It follows from Lemma 6.6.

(2) It follows from the normalizations in Definition 6.5. □

Definition 7.7. Denote by

$$\dot{X}_{K\rtimes\breve{T}}\langle\Psi\rangle\subset X_{K\rtimes\breve{T}}\langle\Psi\rangle$$

the quasi-affine open subscheme where

$$\begin{cases} (x_{\delta,1},\dots,x_{\delta,[F:\mathbb{Q}_p]})\neq 0 & \delta\in\Delta_K-\Psi_0-\Psi_2 \\ (x_{\delta,1},\dots,x_{\delta,[F:\mathbb{Q}_p]+1})\neq 0 & \delta\in\Delta_K\cap\Psi_0-\Psi_2 \\ (x_{\delta,1},\dots,x_{\delta,[F:\mathbb{Q}_p]},s_\delta)\neq 0 & \delta\in\Delta_K\cap\Psi_2-\Psi_0 \\ (x_{\delta,1},\dots,x_{\delta,[F:\mathbb{Q}_p]+1},s_\delta)\neq 0 & \delta\in\Delta_K\cap\Psi_2\cap\Psi_0. \end{cases}$$

Denote by $\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle\subset X_{\breve{B}}$ for the locally closed image (c.f. Remark 7.8) of $\dot{X}_{K\rtimes\breve{T}}\langle\Psi\rangle$.

We call $\{\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle\subset X_{\breve{B}}:(K,\Psi_0,\Psi_2)\}$ the fine stratification of $\mathcal{X}_{\breve{B}}$.

Remark 7.8. For applications, it makes no difference to treat $\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle\subset\mathcal{X}_{\breve{B}}$ as the scheme-theoretic image of $X_{K\rtimes\breve{T}}\langle\Psi\rangle$.

To be more precise, if we write $\overline{\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle\subset X_{\breve{B}}}$ for the scheme-theoretic image of $\dot{X}_{K\rtimes\breve{T}}\langle\Psi\rangle$, then $\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle\subset X_{\breve{B}}$ is $\overline{\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle}-\bigcup_{K'\subsetneq K}\overline{\mathcal{X}_{\breve{B}}[K']\langle\Psi\rangle}$.

Lemma 7.9. *We have*

$$\dim\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle = \dim\dot{X}_{K\rtimes\breve{T}}\langle\Psi\rangle - \#\{\alpha\in\Phi^+-\Phi_K|\alpha\in\Psi_0\} - [k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]\#\Phi_K.$$

Proof. The difference in stabilizers contributes to $\#\{\alpha\in\Phi^+-\Phi_K|\alpha\in\Psi_0\}$. □

7.3. Cone models.

Definition 7.10. The cone model of $X_{K,\check{T}}\langle\Psi\rangle$ is the affine scheme

$$X_{K,\check{T}}\langle\Psi\rangle^{\mathrm{cone}} := \operatorname{Spec}\bar{\mathbb{F}}_p\begin{bmatrix} x_{\alpha,1},\dots,x_{\alpha,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}, s_\alpha & : & \alpha\in\Phi_K \\ t_\delta^{\pm 1} & : & \delta\in\Delta \end{bmatrix}/I^{\mathrm{cone}}$$

where I^{cone} is generated by

$$\left\{\begin{matrix} x_{\alpha,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} & : & \alpha\notin\Psi_0 \\ s_\alpha & : & \alpha\notin\Psi_2 \\ f_\alpha & : & \alpha\in\Psi_2, \mathrm{ht}_K(\alpha)>1 \\ t_\alpha-1 & : & \alpha\in\Psi_0\cup\Psi_2 \end{matrix}\right\}.$$

Here, $f_\alpha = \sum_{\beta\in\Phi_K,\delta\in\Delta_K,\delta+\beta=\alpha} N_{\beta,\delta}\vec{x}_\beta\cdot\vec{x}_\delta$.

Theorem 7.11. *We have*

$$\dim X_{K,\check{T}}\langle\Psi\rangle^{\mathrm{cone}} \geqslant \dim X_{K,\check{T}}\langle\Psi\rangle$$

Proof. Fix the torus coordinates. Consider the bipartite graph whose source vertices are the height-1 roots, whose target vertices are the height-2 roots occurring in the cone equations, and whose edges are

$$\delta\longrightarrow\alpha \qquad\Longleftrightarrow\qquad \alpha-\delta \text{ has height } 1.$$

By Theorem 9.6, successive contractions give a matching containing every target vertex.

Choose the affine chart and the order associated with this matching. First fix all coordinates of the unmatched height-1 roots, together with all coordinates other than $x_{\delta,1}$ for the matched height-1 roots.

In reverse contraction order, the equation attached to the target matched with δ has the form

$$u_\delta x_{\delta,1}+v_\delta=0, \qquad u_\delta\in R^\times,$$

where u_δ and v_δ depend only on coordinates already fixed or previously determined. Thus $x_{\delta,1}$ is determined uniquely. After all matched coordinates have been determined, fix the remaining free height-1 coordinates. This eliminates all height-2 equations. The coordinate blocks belonging to height-2 roots have not been used and remain free.

The rest of the proof follows from the theory of Gröbner basis. We use the height-block elimination order: variables of larger K-height precede every monomial in lower-height variables, and within each height block we use degree-lexicographic order. Thus $x_{\alpha,i}>s_\alpha>x_{\beta,j}>s_\beta$ if $\mathrm{ht}_K(\alpha)>\mathrm{ht}_K(\beta)$. Let $L\subset R$ be the specialized cone ideal and let $I\subset R$ be the specialized ideal of the full equations. The ideal L is generated by homogeneous linear forms (as the height 1 and torus coordinates are fixed). If q is the rank of their coefficient matrix, Gaussian elimination gives independent forms $\ell_1,\dots,\ell_q$ with distinct pivot variables $z_1,\dots,z_q$. Hence

$$\dim R/L=\dim R-q.$$

Apply the same q row operations to the corresponding equations $r_\alpha=f_\alpha+Q_\alpha$. This gives elements $g_1,\dots,g_q\in I$. For a relation of K-height h, the form f_α is linear in the height-$(h-1)$ block and Q_α is independent of that block. Induction down the height blocks therefore gives

$$g_i=z_i+(\text{nonpivot linear terms in the same block})+(\text{terms in lower-height blocks}),$$

so $\mathrm{LT}(g_i) = z_i$. Let $J = (g_1, \ldots, g_q) \subset I$. Since the leading monomials z_i are distinct variables, they are pairwise coprime, and Buchberger's product criterion shows that the g_i form a Gröbner basis for J. Consequently

$$\dim R/J = \dim R - q = \dim R/L.$$

Finally $J \subset I$, so additional equations, including equations produced by dependency rows among the f_α, can only decrease dimension. Thus

$$\dim R/I \leqslant \dim R/J = \dim R/L = \dim R/I^{\mathrm{cone}},$$

which is exactly the desired dimension inequality. No equality of initial ideals and no Gröbner basis for the full ideal I is required. □

8. Example: F_4

In this paper, we use the following notation for roots: for example, if $\Delta = \{\delta_1, \cdots, \delta_4\}$ is the set of simple positive roots, then we denote $\delta_1 + 2\delta_2 + 2\delta_3 + 2\delta_4$ by 1222, and denote by $\delta_2 + \delta_3 + \delta_4$ by 0111. The root system for F_4 is shown in Figure 3.

We have the following theorem:

Theorem 8.1. If $\breve{G} = F_4$, then

$$\dim \mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle^{\mathrm{cone}} \leqslant [F : \mathbb{Q}_p]\#\Phi^+$$

for all fine configurations (K, Ψ_0, Ψ_2). As a consequence, $\mathcal{X}_{\breve{B}}$ is equidimensional of dimension $[F : \mathbb{Q}_p]\#\Phi^+$.

(1) If $F \neq \mathbb{Q}_p$, then the irreducible components of $\mathcal{X}_{\breve{B}}$ are precisely the closure of the strata

$$\mathcal{X}_{\breve{B}}[U]\langle\Psi_0(C), \Psi_2(C)\rangle$$

where $C \subset \mathcal{W}_{\breve{B}}$ is an irreducible component and that

$$\Psi_2(C) = \{\alpha \in \Phi^+ | U_{\alpha^\vee} \text{ is pointwise the cyclotomic Galois character over } C\}$$
$$\Psi_0(C) = \{\alpha \in \Phi^+ | U_{\alpha^\vee} \text{ is pointwise the trivial Galois character over } C\}.$$

(2) If $F = \mathbb{Q}_p$, then in addition to the irreducible components $\overline{\mathcal{X}_{\breve{B}}[U]\langle\Psi_0(C), \Psi_2(C)\rangle}$, $\mathcal{X}_{\breve{B}}$ have 4 extra irreducible components, listed below:

ID	$\Phi^+ - \Phi_K$	Ψ_0	Ψ_2
I	$\{0010\}$	$\{0010\}$	$\{0001, 0011, 0100, 0110, 0120\}$
II	$\{0010\}$	$\{0010\}$	$\{0001, 0011, 0100, 0110, 0120, 1000\}$
III	$\{1000, 0010\}$	$\{0010, 1000\}$	$\left\{\begin{matrix}0001, 0011, 0100, 0110\\ 0120, 1100, 1110, 1120\end{matrix}\right\}$
IV	$\{1000, 0001, 0011, 0010\}$	$\{0001, 0010, 0011, 1000\}$	$\left\{\begin{matrix}0100, 0110, 0111, 0120, 0121, 0122\\ 1100, 1110, 1111, 1120, 1121, 1122\end{matrix}\right\}$

Table 1. Extra components for F_4

In each of the cases listed above, we have $\Psi_2 \subset \Delta_K$ and thus $X_{K\rtimes\breve{T}}\langle\Psi\rangle = X_{K\rtimes\breve{T}}\langle\Psi\rangle^{\mathrm{cone}}$ is smooth.

Proof. Everything is explicitly computable. There are 105 possibilities for K. The maximal cone estimate shows that the inequality is automatic when $[F : \mathbb{Q}_p] \geqslant 2$. For $F = \mathbb{Q}_p$, the 4862 fine pairs give precisely Configurations I–IV. See Appendix A. □

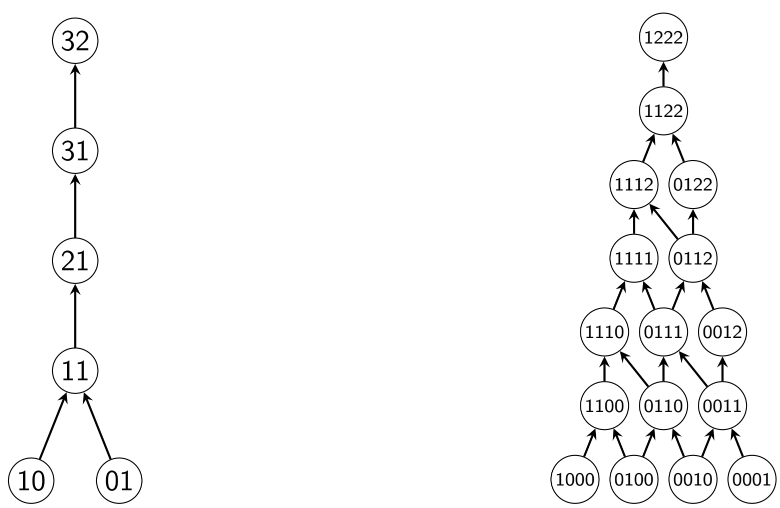
Part II: The combinatorics and the twisted Weil-Deligne stacks

9. Poset graphs

The set of positive roots Φ^+ has the structure of directed graph: two positive roots α,β are connected by a directed edge if and only if $\alpha-\beta\in\Delta$.

Definition 9.1. For each integer h, write $\mathfrak{B}(\Phi^+,\Delta,h)$ for the induced subgraph of Φ^+, consisting of vertices of height either h or $h+1$. So, $\mathfrak{B}(\Phi^+,\Delta,h)$ is a bipartite graph. We will also call $\mathfrak{B}(\Phi^+,\Delta,h)$ the associated bipartite graph of height h.

Example 9.2. Root systems of type A_n, B_n, C_n, and G_2 have a very convenient feature (see Figure 2): their associated bipartite graphs in each height is a tree! In other words, have no loops or cycles.



(A) G_2 (B) B_4/C_4

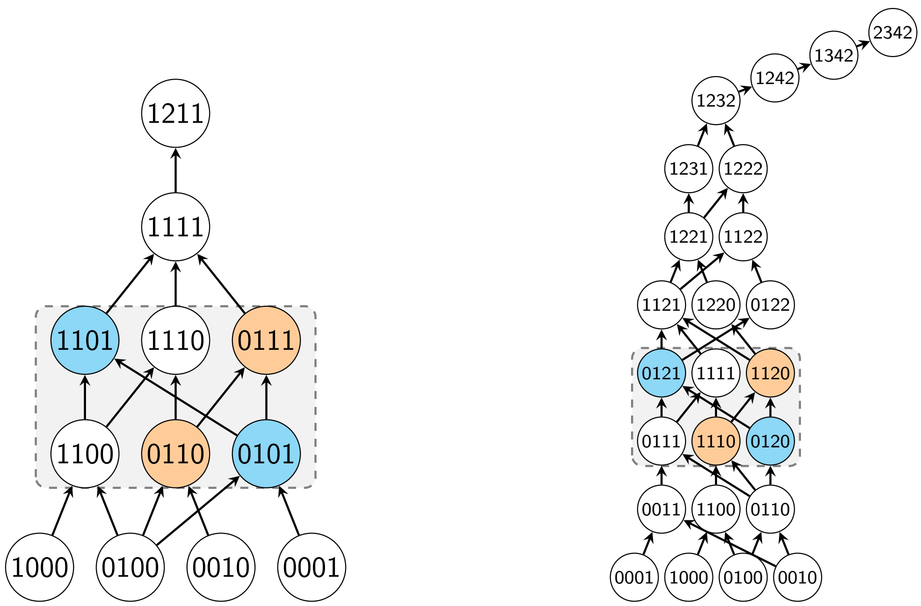
FIGURE 2. Root poset for G_2, B_4 and C_4

Example 9.3. As is illustrated in Figure 3, we see that

$$\mathfrak{B}(\Phi^+_{D_4},\Delta,2)\cong\mathfrak{B}(\Phi^+_{F_4},\Delta,3)$$

are isomorphic bipartite graphs, and they are both isomorphic to the cyclic graph $\vec{C}_6$.

Indeed, we can give an exhaustive list of all cycles in associated bipartite graphs:

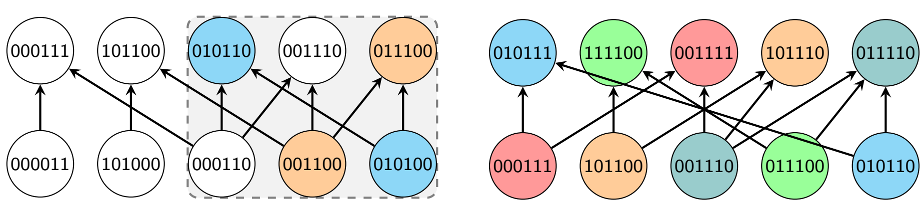


(A) D_4: $\mathfrak{B}(\Phi^+, \Delta, 2) \cong \vec{C}_6$ (B) F_4: $\mathfrak{B}(\Phi^+, \Delta, 3) \cong \vec{C}_6$

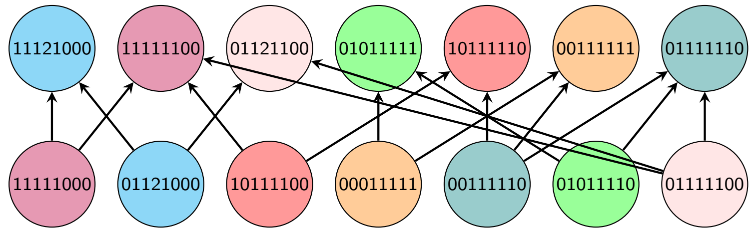
FIGURE 3. Root poset for D_4 and F_4

Height range	F_4	E_6	E_7	E_8
$[2,3]$		$\overrightarrow{C_6}$	$\overrightarrow{C_6}$	$\overrightarrow{C_6}$
$[3,4]$	$\overrightarrow{C_6}$	$\overrightarrow{C_6 \cup_{K_2} C_6}$	$\overrightarrow{C_6 \cup_{K_2} C_6}$	$\overrightarrow{C_6 \cup_{K_2} C_6}$
$[4,5]$			$\overrightarrow{C_6}$	$\overrightarrow{C_6}$
$[5,6]$				$\overrightarrow{C_6 \cup_{K_2} C_6 \cup_{K_2} C_6}$
$[8,9]$			$\overrightarrow{C_6}$	$\overrightarrow{C_6}$
$[9,10]$				$\overrightarrow{C_6 \cup_{K_2} C_6}$
$[14,15]$				$\overrightarrow{C_6}$

TABLE 2. Cycles in associated bipartite graphs.



(A) E_6: $\mathfrak{B}(\Phi^+, \Delta, 2) \supset \vec{C}_6$ (B) E_6: $\mathfrak{B}(\Phi^+, \Delta, 3) \cong \overrightarrow{C_6 \cup_{K_2} C_6}$



(c) E_8: $\mathfrak{B}(\Phi^+, \Delta, 5) \cong \overrightarrow{C_6 \cup_{K_2} C_6 \cup_{K_2} C_6}$

We remind the reader of the following very standard notion:

Definition 9.4. A *bipartite matching* for a bipartite graph, is a collection of edges $\{\beta_i \to \alpha_i | i \in I\}$ such that $\{\alpha_i, \beta_i\}$ consists of exactly $2\#I$ vertices.

The following notion is less standard.

Definition 9.5. Let $\mathfrak{B}$ be a directed graph. We say $\mathfrak{B}$ is *contractible* if there exists a vertex $v \in \mathfrak{B}$ such that there does not exist any edge $v' \to v$ pointing to v, and that there is exactly one edge $v \to v'$ coming out of v, and we say $\mathfrak{B} - \{v, v'\}$ is a contraction of $\mathfrak{B}$.

We say a directed bipartite graph is a cycle if it is connected, not contractible, and non-empty.

Theorem 9.6. *For every reduced root system, the bipartite graph*

$$\mathfrak{B}(\Phi^+, \Delta, 1)$$

contracts to isolated vertices.

Proof. The height-2 roots are the sums of adjacent simple roots. Thus $\mathfrak{B}(\Phi^+, \Delta, 1)$ is the barycentric subdivision of the underlying Dynkin diagram, which is a tree. Successively remove its leaves. $\square$

Proposition 9.7. If Φ is a simple reduced root system, then each $\mathfrak{B}(\Phi^+, \Delta, h)$ contracts to, up to isolated vertices, one of

$$\varnothing,\ \vec{C}_6,\ \overrightarrow{C_6 \cup_{K_1} P_3},\ \overrightarrow{C_6 \cup_{K_2} C_6},\ \overrightarrow{C_6 \cup_{K_2} C_6 \cup_{K_2} C_6}.$$

where P_3 is the chain $\circ \leftarrow \circ \to \circ$ with 3 vertices.

In the next subsection, we deal with situations where $\mathfrak{B}$ is not contractible to points.

9.1. **Admissible bipartite matching.** For each bipartite graph, we will attach to it an algebraic variety that we call the discriminant variety:

Definition 9.8. Let $\mathfrak{B}$ be a directed bipartite graph with source vertices $\mathfrak{B}^{\mathrm{src}} \subset \Phi^+$ and target vertices $\mathfrak{B}^{\mathrm{tar}} \subset \Phi^+$. Set

$$\Delta_{\mathfrak{B}} = \{\alpha - \beta | \alpha \in \mathfrak{B}^{\mathrm{tar}}, \beta \in \mathfrak{B}^{\mathrm{src}}\} \cap \Phi^+,$$

and write $\mathbb{G}_m^{\Delta_{\mathfrak{B}}} = \operatorname{Spec} \bar{\mathbb{F}}_p[x_\delta, x_\delta^{-1} | \delta \in \Delta_{\mathfrak{B}}]$. Note that we don't require $\mathfrak{B}$ to be an induced subgraph of Φ^+, and we only require that the vertices of $\mathfrak{B}$ can be interpreted as positive roots.

The *discriminant variety* $\mathfrak{D}_{\mathfrak{B}}$ is defined as the closed subvariety of

$$\mathbb{G}_m^{\Delta_{\mathfrak{B}}} \times \operatorname{Spec} \bar{\mathbb{F}}_p[y_\beta, \beta \in \mathfrak{B}^{\mathrm{src}}]$$

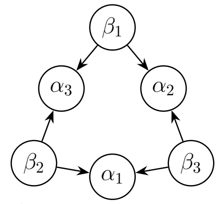
defined by the equations

$$\sum_{\beta \to \alpha} N_{\beta, \alpha - \beta} x_{\alpha - \beta} y_\beta = 0$$

for each $\alpha \in \mathfrak{B}^{\mathrm{tar}}$, where $N_{**} \in \mathbb{Z}_{\neq 0}$ are the standard structure constants defined in Eq. (2).

We say the graph $\mathfrak{B}$ is *admissible* if $\mathfrak{D}_{\mathfrak{B}} \to \mathbb{G}_m^{\Delta_{\mathfrak{B}}}$ is smooth of relative dimension $\#\mathfrak{B}^{\mathrm{src}} - \#\mathfrak{B}^{\mathrm{tar}}$.

Example 9.9. Consider the cyclic graph $\vec{C}_6$:



. Write $\delta_1 = \alpha_3 - \beta_2$, $\delta_2 = \alpha_3 - \beta_1$, and $\delta_3 = \alpha_2 - \beta_1$. If α_* are of height $h+1$ and β_* are of height h, then $\Delta_{\vec{C}_6} = \{\delta_1, \delta_2, \delta_3\} \subset \Delta$. Then the discriminant variety is given by the equations

$$\begin{cases} N_{\beta_2\delta_1} x_{\delta_1} y_{\beta_2} + N_{\beta_1\delta_2} x_{\delta_2} y_{\beta_1} = & 0 \\ N_{\beta_3\delta_2} x_{\delta_2} y_{\beta_3} + N_{\beta_2\delta_3} x_{\delta_3} y_{\beta_2} = & 0 \\ N_{\beta_1\delta_3} x_{\delta_3} y_{\beta_1} + N_{\beta_3\delta_1} x_{\delta_1} y_{\beta_3} = & 0, \end{cases}$$

whose (relative) Jacobian matrix is $\begin{pmatrix} N_{\beta_1\delta_2} x_{\delta_2} & N_{\beta_2\delta_1} x_{\delta_1} & 0 \\ 0 & N_{\beta_2\delta_3} x_{\delta_3} & N_{\beta_3\delta_2} x_{\delta_2} \\ N_{\beta_1\delta_3} x_{\delta_3} & 0 & N_{\beta_3\delta_1} x_{\delta_1} \end{pmatrix}$. So, $\vec{C}_6$ is admissible if and only if $N_{\beta_2\delta_1} N_{\beta_3\delta_2} N_{\beta_1\delta_3} + N_{\beta_1\delta_2} N_{\beta_2\delta_3} N_{\beta_3\delta_1} \neq 0$. Indeed, this value is always equal to one of $\{2, -2, 3\}$ across all root systems, and is non-zero if $p > 3$.

Theorem 9.10. *All induced subgraphs $\mathfrak{B} \subset \mathfrak{B}(\Phi^+, \Delta, h)$ such that $\mathfrak{B}$ contains all height h roots are admissible.*

Proof. For type ABCG, $\mathfrak{B}(\Phi^+, \Delta, h)$ is always a chain, and thus $\mathfrak{D}_{\mathfrak{B}} \to \mathbb{G}_m^{\Delta}$ is not only smooth, but also a vector bundle; by elimination of variables. The remaining types follow from Appendix B. ☐

Corollary 9.11. $\mathcal{X}_{\breve{B}}[U]\langle\Psi\rangle = \dot{\mathcal{X}}_{\breve{B}}\langle\Psi\rangle$ is always smooth of dimension

$$[F : \mathbb{Q}_p]\#\Phi^+ - (\dim \mathrm{span}_{\mathbb{Q}} \Psi_2 - \dim \mathrm{span}_{\mathbb{Q}}(\Psi_2 \cap \Delta)).$$

Write $\mathcal{W}_{\breve{B}}[U]\langle\Psi\rangle \subset \mathcal{W}_{\breve{B}}$ for the image of $\mathcal{X}_{\breve{B}}[U]\langle\Psi\rangle$, then $\mathcal{X}_{\breve{B}}[U]\langle\Psi\rangle \to \mathcal{W}_{\breve{B}}[U]\langle\Psi\rangle$ induces a bijection of irreducible components.

Proof. The Jacobian matrix

$$\left(\frac{\partial r_\alpha}{\partial x_{\beta,1}}\right)_{\alpha \in \Psi_{2,\mathrm{ht}\geqslant 2}, \beta \in \Phi^+}$$

is of full rank by Theorem 9.10. The full-rank Jacobian gives smoothness by [Stacks, Tag 01V9]. On the stratum under consideration every $x_{\delta,1}$ for $\delta \in \Delta$ is invertible, and the full-rank Jacobian minor is a monomial in these variables. Ordered by height, each cone equation is therefore linear in a new variable with unit coefficient. Successive elimination identifies every fiber over $\mathcal{W}_{\breve{B}}[U]\langle\Psi\rangle$ with an affine space of constant dimension. Counting the eliminated variables gives the displayed dimension formula. The morphism is consequently smooth and surjective with geometrically irreducible fibers, and hence induces a bijection on irreducible components. ☐

Corollary 9.12. For each irreducible component $C \subset \mathcal{W}_{\breve{B}}$, $\mathcal{X}_{\breve{B}}(C)$ contains the top-dimensional irreducible component $\overline{\dot{\mathcal{X}}_{\breve{B}}\langle\Psi_0(C), \Psi_2(C)\rangle}$.

On the other hand, If (Ψ_0,Ψ_2) is not of the form $(\Psi_0(C),\Psi_2(C))$, then $\mathcal{X}_{\breve{B}}[U]\langle\Psi\rangle$ is nowhere dense in $\mathcal{X}_{\breve{B}}$.

Proof. Combine Corollary 9.11 and Corollary 6.9. □

Theorem 9.13. If $[F:\mathbb{Q}_p]\geqslant \#\Phi^+ + 2$, then each $\mathcal{X}_{\breve{B}}(C)$ is irreducible. As a consequence, $\mathcal{X}_{\breve{B}}\to\mathcal{W}_{\breve{B}}$ is generically smooth and induces a bijection of irreducible components.

Proof. We always have the trivial upper bound

$$\dim X_{K\rtimes\breve{B}}\langle\Psi\rangle \leqslant ([F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]+1)\#\Phi_K + \dim\breve{T}$$

by Theorem 7.6. So, if $[F:\mathbb{Q}_p]\geqslant\#\Phi^+ + 2$ and $K\neq U$, then $\dim\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle < [F:\mathbb{Q}_p]\#\Phi^+$. The theorem now follows from Corollary 9.12. □

There are only finitely many $F/\mathbb{Q}_p$ cases left, thanks to Theorem 9.13.

9.2. ***K*-restricted posets.** Recall that $K\subset U$ is a $\breve{T}$-stable normal subgroup.

Definition 9.14. We attach to Φ_K the structure of a directed graph: there is a directed edge $\beta\to\alpha$ if and only if $\alpha-\beta\in\Delta_K$.

Similar to Definition 9.1, we write $\mathfrak{B}(\Phi_K,\Delta_K,h)$ for the induced subgraph of Φ_K consisting of vertices of K-height either h or $h+1$.

Example 9.15. Consider $\breve{G}=F_4$, $\Phi_K=\Phi^+-\{0010,0001,0011\}$. The root poset structure of Φ_K is illustrated below (each row has roots of the same K-height, and two vertices are connected with a directed edge if their difference is a root of Δ_K):

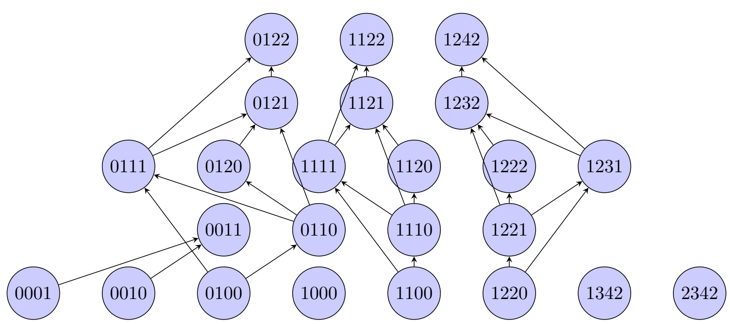


FIGURE 6. The full root poset

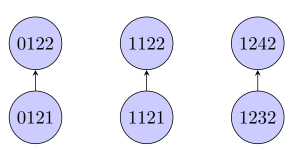


FIGURE 7. The graph $\mathfrak{B}(\Phi_K,\Delta_K,4)$

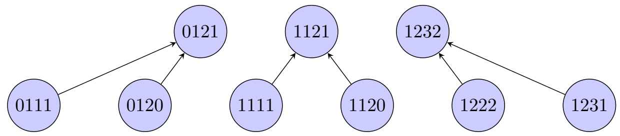


FIGURE 8. The graph $\mathfrak{B}(\Phi_K, \Delta_K, 3)$

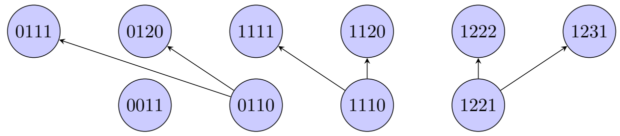


FIGURE 9. The graph $\mathfrak{B}(\Phi_K, \Delta_K, 2)$

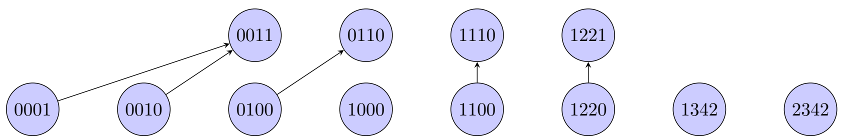


FIGURE 10. The graph $\mathfrak{B}(\Phi_K, \Delta_K, 1)$

As is illustrated in Figure 6, the poset graph Φ_K may have directed edges connecting non-adjacent K-heights. In other words, K-height is not additive. In this paper, edges that skips layers will not be relevant to us anyway, and we may as well redefine the directed tree structure as $\beta \to \alpha$ if and only if $\mathrm{ht}_K(\alpha) - \mathrm{ht}_K(\beta) = 1$ and $\alpha - \beta \in \Delta_K$. All that matter are the bipartite graphs.

10. Twisted Borels and relative root posets

We recall the structure theorem for mod p Galois representations established in our earlier work:

Theorem 10.1. *([Lin22, Theorem 4])* If $\bar{\rho} : \mathrm{Gal}_F \to \breve{G}(\bar{\mathbb{F}}_p)$ is a semisimple Galois representation, then, up to conjugation, there is a Levi subgroup $\breve{M} \subset \breve{G}$ with a maximal torus $\breve{T} \subset \breve{M}$ such that

- the connected centralizer of $\bar{\rho}(I_F)$ in $\breve{M}$ is equal to $\breve{T}$,
- there is an *elliptic Weyl group element* $w \in N_{\breve{M}}(\breve{T})/\breve{T}$ such that $\bar{\rho}$ factors through $\breve{T}(\bar{\mathbb{F}}_p)\langle w \rangle := \langle \breve{T}(\bar{\mathbb{F}}_p), w \rangle \subset \breve{M}$.

Once a Levi $\breve{M}$ is chosen, the maximal torus $\breve{T} \subset \breve{M}$ is uniquely determined.

Proof. The only part not explicitly mentioned in [Lin22, Theorem 4] is the ellipticity of w, which is a consequence of the uniqueness of $\breve{T}$. $\square$

Remark 10.2. In other words, the image of $\bar{\rho}$ is generated by two elements $s \in \breve{T}(\bar{\mathbb{F}}_p)$ and $t \in N_{\breve{M}}(\breve{T})(\bar{\mathbb{F}}_p)$ such that

- s is a *regular semisimple element* of $\breve{M}$, and
- the action of w on the root lattice of $\breve{M}$ does not have eigenvalue 1,
- $tst^{-1} = s^q$, where q is the size of the residue field of F.

Here, s is the image of a generator of the tame quotient of I_F, and t is the image of a Frobenius element.

Note that we embed $N_{\breve{M}}(\breve{T})/\breve{T} \hookrightarrow \breve{M}$ through the canonical Tits representatives (which does not need to be a group homomorphism); indeed, the Tits representatives generate a group of order $2^{k_0}\#(N_{\breve{M}}(\breve{T})/\breve{T})$ for some integer k_0. The order of w in this paper is interpreted as the order of the Tits representative of w (we remark that it is a standard fact that the order of w in $\breve{M}$ and the order of w in $N_{\breve{M}}(\breve{T})/\breve{T}$ are either equal, or differ by a factor of 2).

We don't necessarily have $\breve{T}\langle w\rangle \cong \breve{T} \rtimes \langle w\rangle$.

Setup 10.3. Let $\breve{M} \subset \breve{G}$ be a proper Levi. Let $w \in N_{\breve{M}}(\breve{T})$ be an elliptic Weyl group element.

Let $U_{\breve{M}} := \breve{B} \cap \breve{M}$, which is the unipotent radical of the Borel $\breve{B}_{\breve{M}} = \breve{B} \cap \breve{M}$.

Let $U^{\breve{M}} := \ker(U \to U_{\breve{M}})$, so that $U^{\breve{M}} \rtimes \breve{M}$ is the parabolic subgroup for $\breve{M}$.

Write $\Delta_{\breve{M}} \subset \Delta$ for the simple positive roots of $\breve{M}$.

Write $\Phi_{\breve{M}}$ for the roots of $\breve{M}$.

Write $\Phi^{\breve{M}+} \subset \Phi^+$ for the positive roots that appear in $U^{\breve{M}}$.

Let $\bar{\rho}^{\mathrm{ss}} : \mathrm{Gal}_F \to \breve{T}(\bar{\mathbb{F}}_p)\langle w\rangle$ be an elliptic Galois representation.

Definition 10.4. A *w-relative root* $\alpha = \langle w\rangle\alpha$ is a w-orbit of roots $\{\alpha, w\alpha, w^2\alpha, \ldots\}$. The *$w$-relative root group* is the product

$$U_{\boldsymbol{\alpha}} = U_{\langle w\rangle\alpha} := \prod U_{w^n\alpha^\vee}.$$

The structure theorem imposes the following very strong constraint:

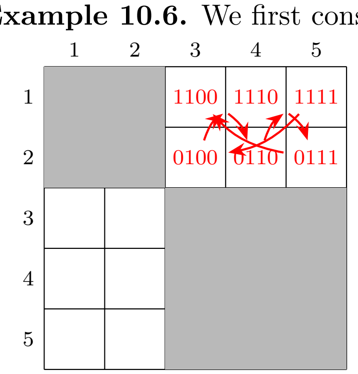
Lemma 10.5. Set $\boldsymbol{\alpha} - \boldsymbol{\alpha} = \{\alpha - \alpha' | \alpha, \alpha' \in \boldsymbol{\alpha}\}$. If $H^2(\mathrm{Gal}_F, U_{\boldsymbol{\alpha}}(\bar{\mathbb{F}}_p)) \neq 0$, then $(\boldsymbol{\alpha} - \boldsymbol{\alpha}) \cap \Phi_{\breve{M}} = \{0\}$.

Proof. Write m for $\#\boldsymbol{\alpha}$ and let F_m/F be the unramified extension of degree m. Then as a Gal_{F_m}-module, $U_{\boldsymbol{\alpha}}(\bar{\mathbb{F}}_p)$ is a direct sum of Galois characters $\chi, \chi^q, \cdots, \chi^{q^{m-1}}$ where $\chi : \mathrm{Gal}_{F_m} \to \bar{\mathbb{F}}_p^\times$ is the adjoint action on $U_{\alpha^\vee}$. If

$$H^2(\mathrm{Gal}_F, U_{\boldsymbol{\alpha}}(\bar{\mathbb{F}}_p)) \cong H^0(\mathrm{Gal}_F, U_{\boldsymbol{\alpha}}(\bar{\mathbb{F}}_p)^\vee(1))^\vee \neq 0,$$

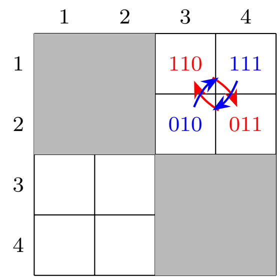
then at least one of $\{\chi, \chi^q, \cdots, \chi^{q^{m-1}}\}$ must be the cyclotomic character, and thus they must all be the cyclotomic character. This implies each quotient χ/χ^{q^k} is the trivial character. If $\boldsymbol{\alpha} - \boldsymbol{\alpha}$ contains a nontrivial root β of $\Phi_{\breve{M}}$, then the action of $\bar{\rho}^{\mathrm{ss}}(I_F)$ on the root group $U_{\beta^\vee}$ is trivial, which contradicts the setup that $\bar{\rho}$ is an elliptic Galois representation (which implies $\bar{\rho}(I_F) \subset \breve{T}$ contains a regular semisimple element of $\breve{M}$). $\square$

Example 10.6. We first consider the case where $\breve{G} = \mathrm{GL}_5$ and that $\breve{M} = \mathrm{GL}_2 \times \mathrm{GL}_3$:



Say $\bar{\rho}^{\mathrm{ss}} = \bar{\rho}_1 \times \bar{\rho}_2$ where $\bar{\rho}_1 : \mathrm{Gal}_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ and $\bar{\rho}_2 : \mathrm{Gal}_F \to \mathrm{GL}_3(\bar{\mathbb{F}}_p)$ are irreducible Galois representations. We have $\mathrm{Hom}_{\mathrm{Gal}_F}(\bar{\rho}_1(1), \bar{\rho}_2) = 0$ as they are irreducible Galois representations of different rank. The single six-element w-orbit is drawn above. Note that $1111 - 1100 = 0011$, which is a root for $\mathrm{GL}_3 \subset \check{M}$.

On the other hand, consider $\check{G} = \mathrm{GL}_4$:



There are two w-orbits: $\{110, 011\}$ and $\{111, 010\}$. The difference $110 - 011$ is not even a root. Neither is the difference $111 - 010 = 101$. This shows that both

$$H^2(\mathrm{Gal}_F, U_{\langle w \rangle 110}(\bar{\mathbb{F}}_p)) \text{ and } H^2(\mathrm{Gal}_F, U_{\langle w \rangle 010}(\bar{\mathbb{F}}_p))$$

can be non-trivial. Nevertheless, they cannot be non-trivial *simultaneously*, as $110 - 010 = 100$ is a root of $\check{M}$. This observation can also be confirmed by local Tate duality – nonvanishing of $H^2(\mathrm{Gal}_F, U^{\check{M}}(\bar{\mathbb{F}}_p))$ implies $\bar{\rho}_1(1) \cong \bar{\rho}_2$, and by Schur's lemma, we have $\mathrm{Hom}_{\mathrm{Gal}_F}(\bar{\rho}_1(1), \bar{\rho}_2) \cong \bar{\mathbb{F}}_p$.

Lemma 10.7. (1) If $H^2(\mathrm{Gal}_F, U_{\alpha}(\bar{\mathbb{F}}_p))$ and $H^2(\mathrm{Gal}_F, U_{\alpha'}(\bar{\mathbb{F}}_p))$ are both nontrivial, then

$$((\alpha \cup \alpha') - (\alpha \cup \alpha')) \cap \Phi_{\check{M}} = \{0\}.$$

(2) If $H^i(\mathrm{Gal}_F, U_{\alpha}(\bar{\mathbb{F}}_p)) \neq 0$ for $i = 0, 2$, then it is equal to $\bar{\mathbb{F}}_p$.

Proof. (1) Clear. (2) It follows from Frobenius reciprocity. □

Definition 10.8. Let $\alpha = \sum_{\delta \in \Delta} n_\delta \delta \in \Phi^+$. Define

$$\mathrm{ht}^{\check{M}}(\alpha) := \sum_{\delta \notin \Delta_{\check{M}}} n_\delta, \text{ and } \mathrm{ht}_{\check{M}}(\alpha) := \sum_{\delta \in \Delta_{\check{M}}} n_\delta.$$

The descending central filtration for $U^{\check{M}}$ coincides with the $\mathrm{ht}^{\check{M}}$ filtration: if we write

$$\mathrm{Lie}\, U_k^{\check{M}} = \begin{cases} \mathrm{Lie}\, U^{\check{M}} & k = 1 \\ [\mathrm{Lie}\, U^{\check{M}}, \mathrm{Lie}\, U_{k-1}^{\check{M}}] & k > 1, \end{cases}$$

then

$$\bar{U}_k^{\check{M}} := U_k^{\check{M}} / U_{k+1}^{\check{M}} = \prod_{\alpha \in \Phi^{\check{M}+}, \mathrm{ht}^{\check{M}}(\alpha) = k} U_{\alpha^\vee}.$$

Since w preserves the height $\mathrm{ht}^{\check{M}}$, each $U_k^{\check{M}}$ is a product of w-relative root groups.

10.1. **Relative root posets.** Recall that Φ^+ is a directed graph, whose directed edges $\beta \to \alpha$ are precisely positive roots α, β such that $\alpha - \beta \in \Delta$. We can similarly equip $\Phi^{\check{M}+}$ with a directed graph structure whose directed edges $\beta \to \alpha$ are positive roots α, β satisfying $\alpha - \beta \in \Phi^{\check{M}+}$ and $\mathrm{ht}^{\check{M}}(\alpha - \beta) = 1$.

Definition 10.9. The *w-root poset*

$$\mathbb{\Phi}^+ = \mathbb{\Phi}^{\check{M}+} := \Phi^{\check{M}+}/w$$

is by definition the quotient graph of $\Phi^{\check{M}+}$ by the cyclic group action of $\{1, w, w^2, \cdots\}$. Write

$$\mathbb{\Delta} = \mathbb{\Delta}^{\check{M}} := \{\mathbb{\alpha} \in \mathbb{\Phi}^+ |\, \mathrm{ht}^{\check{M}}(\mathbb{\alpha}) = 1\},$$

and call it the *w-relative simple roots.*

Denote by $\mathfrak{B}(\mathbb{\Phi}^+, \mathbb{\Delta}, h) \subset \mathbb{\Phi}^+$ for the induced bipartite subgraph consisting of w-relative roots $\mathbb{\alpha}$ with $\mathrm{ht}^{\check{M}}(\mathbb{\alpha}) \in \{h, h+1\}$.

We say a w-relative root $\mathbb{\alpha}$ is *non-obstructing* if $(\mathbb{\alpha} - \mathbb{\alpha}) \cap \Phi_{\check{M}} \neq \{0\}$. We say a set S of w-relative root is *non-obstructing* if $(\bigcup_{\mathbb{\alpha} \in S} \mathbb{\alpha} - \bigcup_{\mathbb{\alpha} \in S} \mathbb{\alpha}) \cap \Phi_{\check{M}} \neq \{0\}$.

We say an induced bipartite subgraph $\mathfrak{B} \subset \mathbb{\Phi}^+$ is *non-obstructing* if the set of target vertices $\mathfrak{B}^{\mathrm{tar}}$ is non-obstructing.

Definition 10.10. Let $\mathfrak{B} \subset \mathbb{\Phi}^+$ be an induced bipartite subgraph. The *associated absolute bipartite graph* $\mathfrak{B}_{\mathrm{abs}}$ is by definition the directed bipartite graph such that

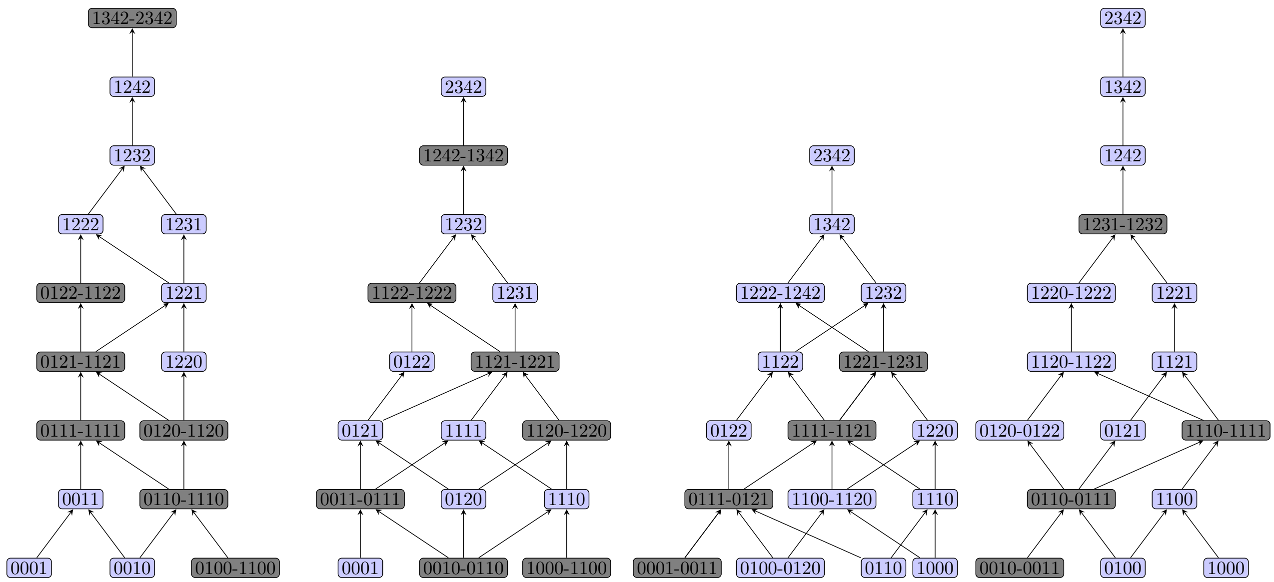
- the source vertices of $\mathfrak{B}_{\mathrm{abs}}$ are the absolute roots contained in the w-relative roots that are source vertices of $\mathfrak{B}$,
- the target vertices $\mathfrak{B}_{\mathrm{abs}}$ are the absolute roots contained in the w-relative roots that are target vertices of $\mathfrak{B}$,
- the graph structure of $\mathfrak{B}_{\mathrm{abs}}$ is given by the induced subgraph structure of Φ^+.

Theorem 10.11. (1) If $\check{G} = F_4$, then all cycles in $\mathfrak{B}(\mathbb{\Phi}^+, \mathbb{\Delta}, h)$ are non-obstructing.

(2) If $\check{G} = E_6, E_7$ and E_8, then all cycles $\mathfrak{C}$ in $\mathfrak{B}(\mathbb{\Phi}^+, \mathbb{\Delta}, h)$ are either non-obstructing or have admissible associated absolute bipartite graph $\mathfrak{C}_{\mathrm{abs}}$.

Proof. This follows from Appendix B. □

Example 10.12. (w-root poset graphs for F_4) We will only show the rank 1 Levi cases here: there are four rank 1 Levi's, corresponding to the four simple roots; and each rank 1 Levi has a unique elliptic Weyl group element. The non-obstructing w-relative roots are colored by gray.



(A) w-root poset for $\breve{M}_{1000}$ (B) w-root poset for $\breve{M}_{0100}$ (C) w-root poset for $\breve{M}_{0010}$ (D) w-root poset for $\breve{M}_{0001}$

11. Unramified descent

Let F_n/F be the unramified extension of p-adic fields of degree n coprime to p. Then $\mathbf{E}_{F_n}/\mathbf{E}_F$ is also an unramified extension. As a consequence, $\mathbf{E}_{F_n}/\mathbf{E}_F$ is a finite étale extension. Let A be an $\bar{\mathbb{F}}_p$-algebra.

Lemma 11.1. *$\mathbf{E}_{F_n}/\mathbf{E}_F$ is necessarily an unramified extension of degree n.*

Proof. Recall that $\Gamma_F = \mathrm{Gal}_{F(\mu_{p^\infty})/F} \cong \mathbb{Z}_p \times \Delta_F$, and $\mathbf{E}_F = (k_{F(\mu_{p^\infty})}((T'_F)))^{\Delta_F} = k_{F(\mu_{p^\infty})^{\Delta_F}}((T_F))$. Here, k_F is the residual field of F. Note that $F(\mu_{p^\infty})^{\Delta_F}/F$ is a p-power extension, and thus $F_n(\mu_{p^\infty})^{\Delta_F}/F(\mu_{p^\infty})^{\Delta_F}$ is unramified of degree n. $\square$

Let M be an étale (φ,Γ)-module over $\mathbf{E}_{F,A}$. Then $M \otimes_{\mathbf{E}_{F,A}} \mathbf{E}_{F_n,A}$ is an étale (φ,Γ)-module over $\mathbf{E}_{F_n,A}$, together with Galois descent data:

Proposition 11.2. Write σ for a generator of $\mathrm{Gal}(\mathbf{E}_{F_n}/\mathbf{E}_F)$. There is an equivalence of categories

$$\{\text{étale } (\varphi,\Gamma)\text{-modules } (M,\phi_M,\gamma_M) \text{ over } \mathbf{E}_{F,A}\} \cong \left\{\begin{array}{c}\text{étale } (\varphi,\Gamma)\text{-modules } (M',\phi_{M'},\gamma_{M'}) \text{ over } \mathbf{E}_{F_n,A} \\ \text{together with } \sigma_{M'}: \sigma^*M' \xrightarrow{\cong} M' \\ \text{that commutes with } \phi_{M'} \text{ and } \gamma_{M'}, \text{ and satisfies the cyclic cocy}\end{array}\right.$$

Proof. Because $\mathbf{E}_{F_n}/\mathbf{E}_F$ is a finite étale Galois extension with Galois group $\mathrm{Gal}(\mathbf{E}_{F_n}/\mathbf{E}_F)$, standard Galois descent establishes the equivalence at the level of underlying modules. Furthermore, since F_n/F is unramified, the action of $\mathrm{Gal}(\mathbf{E}_{F_n}/\mathbf{E}_F)$ commutes with both the absolute Frobenius φ and the cyclotomic Galois group Γ. Thus, the descent datum naturally respects the (φ,Γ)-structure. $\square$

Corollary 11.3. Let M be an étale (φ,Γ)-module over $\mathbf{E}_{F,A}$. Write $M' := M \otimes_{\mathbf{E}_F} \mathbf{E}_{F_n}$.

(1) The map $c \mapsto c \otimes 1$ induces an embedding

$$Z^1_{\mathrm{Herr}}(M) \hookrightarrow Z^1_{\mathrm{Herr}}(M').$$

(2) $\sigma_{M'}$ induces a semilinear action of $\mathrm{Gal}(\mathbf{E}_{F_n}/\mathbf{E}_F)$ on $Z^1_{\mathrm{Herr}}(M')$ such that $Z^1_{\mathrm{Herr}}(M)$ is identified with the fixed points.

(3) The trace $\mathrm{tr}_{\sigma_{M'}} = \frac{1}{[\mathbf{E}_{F_n}:\mathbf{E}_F]}\sum_i \sigma^i_{M'}$ induces a section

$$Z^1_{\mathrm{Herr}}(M') \to Z^1_{\mathrm{Herr}}(M)$$

of the embedding in part (1).

Proof. We have $C^\bullet_{\mathrm{Herr}}(M') = C^\bullet_{\mathrm{Herr}}(M) \otimes_{\mathbf{E}_F} \mathbf{E}_{F_n}$. The map $c \mapsto c \otimes 1$ respects d^1_{Herr}, proving (1). Because the semilinear action of $\sigma_{M'}$ on M' commutes with φ and Γ, it extends to the Herr complex. The fixed points of $Z^1_{\mathrm{Herr}}(M')$ under this finite group action recover exactly $Z^1_{\mathrm{Herr}}(M)$, establishing (2). Finally, because the degree $[\mathbf{E}_{F_n} : \mathbf{E}_F]$ divides n, which is coprime to p, it is invertible in the coefficient field $\bar{\mathbb{F}}_p$, making the trace map in (3) a well-defined idempotent projecting $Z^1_{\mathrm{Herr}}(M')$ onto $Z^1_{\mathrm{Herr}}(M)$. □

Remark 11.4. Since $\mathrm{Gal}_F \to \mathrm{Gal}(F_n/F)$ has no (tautological) splitting, we do not have an analogue of Corollary 11.3 for Galois cohomology. The action of $\mathrm{Gal}(F_n/F)$ on $H^1(\mathrm{Gal}_{F_n}, \mathrm{Res}_{F_n/F} V)$ does not lift to an action of $\mathrm{Gal}(F_n/F)$ on cocycles $Z^1(\mathrm{Gal}_{F_n}, \mathrm{Res}_{F_n/F} V)$.

The non-liftability of the $\mathrm{Gal}(F_n/F)$-action to cocycles indeed carries over to the setup of Herr complex cohomology. There is no *linear* action of $\mathrm{Gal}(F_n/F)$ on $Z^1_{\mathrm{Herr}}(M')$ lifting the action on $H^1_{\mathrm{Herr}}(M')$: for any (hypothetical) linear action $\tilde{\sigma}$, the power $\tilde{\sigma}^n$ will never be the identity map, rather $\tilde{\sigma}^n - \mathrm{id}$ is a nontrivial coboundary. If one systematically corrects this coboundary, one will always end up with a semilinear action. Such a correction is impossible to do for Galois cocycles, as the coefficient module V is not large enough to absorb the twist.

One of the advantages of the Herr complex is that it enables us to lift linear (inflation-restriction) Galois actions on cohomology groups to semilinear actions on cocycle groups.

Theorem 11.5. Let $\bar{\chi} : \mathrm{Gal}_F \to \bar{\mathbb{F}}_p^\times$ be a Galois character, and let $\bar{\chi}_n$ denote its restriction to Gal_{F_n}. Let $\bar{\chi}_{\mathrm{cyc}}$ denote the mod p cyclotomic character. The full cohomology group $H^1(\mathrm{Gal}_{F_n}, \bar{\chi}_n)$ decomposes as an $\bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]$-module as follows:

$$H^1(\mathrm{Gal}_{F_n}, \bar{\chi}_n) \cong \begin{cases} \bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]}, & \bar{\chi}_n \not\cong \bar{\mathbb{F}}_p,\ \bar{\mathbb{F}}_p(1), \\ \bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]} \oplus \bar{\chi}, & \bar{\chi}_n \cong \bar{\mathbb{F}}_p,\ \bar{\chi}_n \not\cong \bar{\mathbb{F}}_p(1), \\ \bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]} \oplus (\bar{\chi} \otimes \bar{\chi}_{\mathrm{cyc}}^{-1}), & \bar{\chi}_n \cong \bar{\mathbb{F}}_p(1),\ \bar{\chi}_n \not\cong \bar{\mathbb{F}}_p, \\ \bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]} \oplus \bar{\chi} \oplus (\bar{\chi} \otimes \bar{\chi}_{\mathrm{cyc}}^{-1}), & \bar{\chi}_n \cong \bar{\mathbb{F}}_p \cong \bar{\mathbb{F}}_p(1). \end{cases}$$

Proof. Since the order of $\mathrm{Gal}(F_n/F)$ is prime to p, the group algebra $\bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]$ is split semisimple. A representation over this algebra is entirely determined by the dimensions of its isotypic components.

Let $\vartheta : \mathrm{Gal}(F_n/F) \to \bar{\mathbb{F}}_p^\times$ be a character, which we inflate to Gal_F. Because $H^i(\mathrm{Gal}(F_n/F), -) = 0$ for $i > 0$, the inflation–restriction exact sequence yields an isomorphism of the ϑ-isotypic component:

$$H^1(\mathrm{Gal}_{F_n}, \bar{\chi}_n)_\vartheta \cong H^1(\mathrm{Gal}_F, \bar{\chi} \otimes \vartheta^{-1}).$$

The local Euler characteristic formula states that

$$\dim H^1(\mathrm{Gal}_F, \bar{\chi} \otimes \vartheta^{-1}) = [F : \mathbb{Q}_p] + \dim H^0(\mathrm{Gal}_F, \bar{\chi} \otimes \vartheta^{-1}) + \dim H^0(\mathrm{Gal}_F, (\bar{\chi} \otimes \vartheta^{-1})^\vee(1)).$$

Because the character $\bar{\chi}\otimes\vartheta^{-1}$ is one-dimensional, each H^0-term is either 0 or 1. Specifically, $\dim H^0(\mathrm{Gal}_F,\bar{\chi}\otimes\vartheta^{-1})=1$ if and only if $\bar{\chi}\otimes\vartheta^{-1}\cong\bar{\mathbb{F}}_p$, which occurs exactly when $\vartheta=\bar{\chi}$. Note that this case is possible if and only if $\bar{\chi}_n\cong\bar{\mathbb{F}}_p$.

Similarly, $\dim H^0(\mathrm{Gal}_F,(\bar{\chi}\otimes\vartheta^{-1})^\vee(1))=1$ if and only if $\bar{\chi}\otimes\vartheta^{-1}\cong\bar{\mathbb{F}}_p(1)$, meaning $\vartheta=\bar{\chi}\otimes\bar{\chi}_{\mathrm{cyc}}^{-1}$. This case is possible if and only if $\bar{\chi}_n\cong\bar{\mathbb{F}}_p(1)$.

For any character ϑ not meeting these conditions, the H^0 terms vanish, giving a baseline dimension of $[F:\mathbb{Q}_p]$. The regular representation $\bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]$ contains exactly one copy of every character ϑ. Thus, the uniform dimension of $[F:\mathbb{Q}_p]$ across all isotypic components contributes precisely $\bar{\mathbb{F}}_p[\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]}$ to the overall structure.

The only deviations from this regular baseline are the extra dimensions arising from non-zero H^0 terms. If $\bar{\chi}_n\cong\bar{\mathbb{F}}_p$, the cohomology group gains one extra copy of the character $\vartheta=\bar{\chi}$. If $\bar{\chi}_n\cong\bar{\mathbb{F}}_p(1)$, it gains one extra copy of the character $\vartheta=\bar{\chi}\otimes\bar{\chi}_{\mathrm{cyc}}^{-1}$. Summing the baseline regular representation and these specific anomalous one-dimensional pieces yields the four cases. $\square$

Corollary 11.6. Write $M'=M'_{\bar{\chi}_n}$ for the rank 1 étale (φ,Γ)-module over $\mathbf{E}_{F_n,\bar{\mathbb{F}}_p[t^{\pm1}]}$ that is the universal unramified twist of $\bar{\chi}_n$. Then, we have

$$H^1_{\mathrm{Herr}}(M')\cong\begin{cases}\bar{\mathbb{F}}_p[t^{\pm1}][\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]} & \bar{\chi}_n\neq\bar{\mathbb{F}}_p\\ \bar{\mathbb{F}}_p[t^{\pm1}][\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]}\oplus\bar{\chi} & \bar{\chi}_n=\bar{\mathbb{F}}_p.\end{cases}$$

So,

$$H^1_{\mathrm{Herr},+}(M')\cong\bar{\mathbb{F}}_p[t^{\pm1}][\mathrm{Gal}(F_n/F)]^{\oplus[F:\mathbb{Q}_p]}\oplus(\bar{\mathbb{F}}_p[t^{\pm1}]\bar{\chi}\otimes_{\mathbb{F}_p}k_{F_{n,\mathrm{cyc}}})$$

as a $\bar{\mathbb{F}}_p[t^{\pm1}][\mathrm{Gal}(F_n/F)]$-module in both cases.

Proof. By Lemma 2.2, $H^1_{\mathrm{Herr}}(M')$ does not have the cyclotomic terms, compared to Theorem 11.5. When $\bar{\chi}_n$ is trivial, there is the unramified cocycle $(1,0)\in Z^1_{\mathrm{Herr}}(M')$ (c.f. Subsection 2.3), which is a coboundary over the locus $(t\neq1)$. Quotienting out by $[(1,0)]$, we obtain a finite free $\bar{\mathbb{F}}_p[t^{\pm1}]$-module whose isotypic parts have the same rank by Theorem 11.5.

Write $H^1_{\mathrm{Herr}}(M')_\vartheta$ for the ϑ-isotypic part of the (maximal) finite free quotient $H^1_{\mathrm{Herr,tf}}(M')$ of $H^1_{\mathrm{Herr}}(M')$. Choose a basis $\{v_{k,\vartheta}\}_{k=1}^{[F:\mathbb{Q}_p]}$ for each isotypic part. Set

$$c'_k:=\sum_\vartheta v_{k,\vartheta},\ k=1,\dots,[F:\mathbb{Q}_p].$$

Note that we have the projection operators

$$e_\vartheta=\frac{1}{n}\sum_{\sigma\in\mathrm{Gal}(F_n/F)}\vartheta(\sigma)^{-1}\,\sigma$$

satisfying

$$e_\vartheta(c'_k)=v_{k,\vartheta}.$$

The upshot is that

$$H^1_{\mathrm{Herr,tf}}(M')=\bigoplus_{k=1}^{[F:\mathbb{Q}_p]}\bar{\mathbb{F}}_p[t^{\pm1}][\mathrm{Gal}(F_n/F)]c'_k.$$ $\square$

Note that

$$k_{F_n,\mathrm{cyc}}\cong\mathbb{F}_p[\mathrm{Gal}(k_{F_{n,\mathrm{cyc}}}/\mathbb{F}_p)]\cong\mathbb{F}_p[\mathrm{Gal}(F_n/F)]^{\oplus[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$$

by the normal basis theorem. Thus

$$H^1_{\mathrm{Herr},+}(M') \cong \bar{\mathbb{F}}_p[t^{\pm 1}][\mathrm{Gal}(F_n/F)]^{\oplus [F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}$$

as a $\bar{\mathbb{F}}_p[t^{\pm 1}][\mathrm{Gal}(F_n/F)]$-module. So, we get the following corollary:

Corollary 11.7. There exist cocycles

$$c'_1, \cdots, c'_{[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]} \in Z^1_{\mathrm{Herr},+}(M')$$

such that

$$\{\sigma^j \cdot c'_i | \sigma^j \in \mathrm{Gal}(F_n/F), 1 \leqslant i \leqslant [F:\mathbb{Q}_p] + [k_{F_{\mathrm{cyc}}} : \mathbb{F}_p]\}$$

forms a basis for $H^1_{\mathrm{Herr},+}(M')$. So,

$$\{c_i := \sum_{j=0}^{n-1} \sigma^j c'_i | i = 1, \cdots, [F:\mathbb{Q}_p] + [k_{F_{\mathrm{cyc}}} : \mathbb{F}_p]\}$$

forms a basis for $H^1_{\mathrm{Herr},+}(M)$.

Example 11.8. Let $F = \mathbb{Q}_p$ and $F_2 = \mathbb{Q}_{p^2}$.

Let $\chi : \mathrm{Gal}_{F_2} \to \bar{\mathbb{F}}_p^\times$ be a character of order $p^2 - 1$. For arbitrary scalars $a, b \neq 0 \in \bar{\mathbb{F}}_p^\times$, we can construct an irreducible Galois representation $\rho_{a,b} : \mathrm{Gal}_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ such that $\rho_{a,b}|_{\mathrm{Gal}_{F_2}} = \begin{pmatrix} \chi & \\ & \chi^p \end{pmatrix}$ and $\rho_{a,b}(\mathrm{Frob}) = \begin{pmatrix} & a \\ b & \end{pmatrix}$. We form a 3-dimensional Galois representation $\left(\begin{array}{c|c} \rho_{a,b} & \begin{matrix} c_1 \\ c_2 \end{matrix} \\ \hline & 1 \end{array}\right)$ We remark that the conjugation action of $\rho_{a,b}(\mathrm{Frob})$ on $\mathrm{Ext}^1(\bar{\mathbb{F}}_p, \rho_{a,b})$ is not necessarily idempotent. Indeed, $\rho_{a,b}(\mathrm{Frob}^2)$ acts by scalar multiplication by ab.

Next, we convert the Galois representations to étale (φ, Γ)-modules. Write $D_F(-)$ for the Fontaine functor from Galois representations to étale (φ, Γ)-modules over $\mathbf{E}_F$. Then $D_F(\rho_{a,b})$ is an étale (φ, Γ)-module over $\mathbf{E}_{F,\bar{\mathbb{F}}_p} = \mathbf{E}_{F,\bar{\mathbb{F}}_p}$. We have

$$D_{F_2}(\rho_{a,b}|_{\mathrm{Gal}_{F_2}}) = D_F(\rho_{a,b}) \otimes_{\mathbf{E}_{F,\bar{\mathbb{F}}_p}} \mathbf{E}_{F_2,\bar{\mathbb{F}}_p} = D_F(\rho_{a,b}) \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2},$$

and is equipped with Galois descent data $\tau : x \otimes y (\in D_F(\rho_{a,b}) \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \mapsto x \otimes y^p$. The descent data τ is always idempotent: $\tau^2 = 1$.

Over the unramified extension $F_2 = \mathbb{Q}_{p^2}$, the restricted Galois representation decomposes as a direct sum of the characters χ and χ^p. Correspondingly, the étale (φ, Γ)-module splits as $M' := D_{F_2}(\rho_{a,b}|_{\mathrm{Gal}_{F_2}}) = M'_\chi \oplus M'_{\chi^p}$. The τ-action on

$$Z^1_{\mathrm{Herr}}(M') = Z^1_{\mathrm{Herr}}(M'_\chi) \oplus Z^1_{\mathrm{Herr}}(M'_{\chi^p})$$

exchanges the two isotypic factors.

Let $\{\delta_1, \delta_2, \gamma = \delta_1 + \delta_2\}$ be the positive roots of GL_3. We have

$$\begin{aligned} M'_\chi &= U_{\gamma^\vee}(\mathbf{E}_{F_2,\bar{\mathbb{F}}_p}) \\ M'_{\chi^p} &= U_{\delta_2^\vee}(\mathbf{E}_{F_2,\bar{\mathbb{F}}_p}) \end{aligned}$$

So, if

$$\{c_{\delta_2,1}, c_{\delta_2,2}, c_{\delta_2,3}, c_{\delta_2,4}\} \subset Z^1_{\mathrm{Herr},+}(U_{\delta_2^\vee}(\mathbf{E}_{F_2,\bar{\mathbb{F}}_p[a^{\pm 1}, b^{\pm 1}]}))$$

forms a basis for $H^1_{\mathrm{Herr},+}(U_{\delta_2^\vee}(\mathbf{E}_{F_2,\bar{\mathbb{F}}_p[a^{\pm 1},b^{\pm 1}]}))$, then

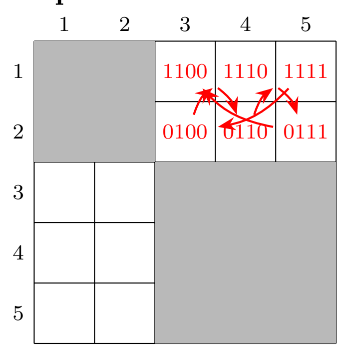
$$\{\tau c_{\delta_2,1}, \tau c_{\delta_2,2}, \tau c_{\delta_2,3}, \tau c_{\delta_2,4}\} \subset Z^1_{\mathrm{Herr},+}(U_{\gamma^\vee}(\mathbf{E}_{F_2,\bar{\mathbb{F}}_p[a^{\pm 1},b^{\pm 1}]}))$$

forms a basis for $H^1_{\mathrm{Herr},+}(U_{\gamma^\vee}(\mathbf{E}_{F_2,\bar{\mathbb{F}}_p[a^{\pm 1},b^{\pm 1}]}))$. Moreover,

$$\{(c_{\delta_2,1}, \tau c_{\delta_2,1}), (c_{\delta_2,2}, \tau c_{\delta_2,2}), (c_{\delta_2,3}, \tau c_{\delta_2,3}), (c_{\delta_2,4}, \tau c_{\delta_2,4})\} \subset Z^1_{\mathrm{Herr},+}((U_{\delta_2^\vee} \oplus U_{\gamma^\vee})(\mathbf{E}_{F,\bar{\mathbb{F}}_p[a^{\pm 1},b^{\pm 1}]}))$$

forms a basis for $H^1_{\mathrm{Herr},+}((U_{\delta_2^\vee} \oplus U_{\gamma^\vee})(\mathbf{E}_{F,\bar{\mathbb{F}}_p[a^{\pm 1},b^{\pm 1}]}))$. Here, we always choose $\{c_{\delta_2,3}, c_{\delta_2,4}\}$ to be the unramified cocycles.

Example 11.9. Now consider $\check{G} = \mathrm{PGL}_5$:



Consider the Galois representations of the form

$$\rho|_{I_n} = \begin{pmatrix} \chi & & * & * & * \\ & \chi^p & * & * & * \\ & & \omega & & \\ & & & \omega^p & \\ & & & & \omega^{p^2} \end{pmatrix}, \qquad \rho(\mathrm{Frob}) = \begin{pmatrix} & a_1 & * & * & * \\ a_2 & & * & * & * \\ & & & a_3 & \\ & & & & a_4 \\ & & a_5 & & \end{pmatrix}.$$

Set $R = \bar{\mathbb{F}}_p[a_i^{\pm 1} : i = 1, 2, 3, 4, 5]$. The order of w is 6. So, assume $F = \mathbb{Q}_p$ and $F_6 = \mathbb{Q}_{p^6}$. There is one w-relative root containing the six positive cross-block roots:

$$\alpha = \{1100, 0110, 1111, 0100, 1110, 0111\}.$$

Note that F_6 is the splitting field of this relative root. Let $c_{1100,1}, \ldots, c_{1100,m}$ be cocycle representatives for an R-basis of $H^1_{\mathrm{Herr},+}(U_{1100}(\mathbf{E}_{F_6,R}))$. Write σ for a generator of $\mathrm{Gal}(F_6/F)$ and τ for the induced semilinear action on U_α. By Shapiro descent, the universal cocycle for $Z^1_{\mathrm{Herr},+}(U_\alpha(\mathbf{E}_{F,R}))$ is

$$c_\alpha^{\mathrm{univ}} = \sum_{i=1}^{m} x_i \sum_{j=0}^{5} \tau^j c_{1100,i} \in Z^1_{\mathrm{Herr},+}(U_\alpha(\mathbf{E}_{F,R[x_1,\ldots,x_m]})).$$

12. Twisted destackification and twisted Weil-Deligne stacks

Setup 10.3 is in force in this subsection. We also keep the notation of the previous section. Let $\mathbb{K} \subset U^{\check{M}}$ be a $\check{T}\langle w\rangle$-stable subgroup. Write $\mathbb{K} = \prod_{\alpha \in \Phi_{\mathbb{K}}} U_\alpha$. Write $\Delta_{\mathbb{K}} \subset \Phi_{\mathbb{K}}$ for the set of relative roots occurring in $\mathrm{Lie}\,\mathbb{K}/[\mathrm{Lie}\,\mathbb{K}, \mathrm{Lie}\,\mathbb{K}]$. Write n for the order of w in the Weyl group.

Lemma 12.1. *If $p > h_{\check{G}}$, then $p \nmid (\# N_{\check{G}}(\check{T})/\check{T})$.*

Proof. It is a standard fact. See, for example, [Hum90, Theorem 3.9]. $\square$

As a consequence, $p \nmid n$.

12.1. **Destackification of $\mathcal{X}_{\check{T}\langle w\rangle}$.** We have

$$\mathcal{X}_{\check{T}\langle w\rangle} \cong \coprod [\check{T}/_w\check{T}]$$

is a disjoint union of $[\check{T}/_w\check{T}]$. Unlike the untwisted case, the $\check{T}$-action on $\check{T}$ is non-trivial. We set $X_{\check{T}\langle w\rangle} := \coprod \check{T}$.

12.2. **Destackification of $\mathcal{X}_{\mathbb{K}\rtimes\check{T}\langle w\rangle}$.** We fix a total ordering on $\Phi_{\mathbb{K}}$ compatible with $\mathbb{K}$-height.

We recursively build destackifications $X_{\leqslant\boldsymbol{\alpha}}$ for $\mathcal{X}_{\leqslant\boldsymbol{\alpha}} := \mathcal{X}_{U_{\leqslant\boldsymbol{\alpha}}\rtimes\check{T}\langle w\rangle}$ in a manner similar to the procedure of Subsection 3.1, with the only difference being that sections $\xi : P_A \to P'_A$ are required to be both Γ-stable and $\mathrm{Gal}(F_n/F)$-stable.

Proposition 12.2. The morphism $X_{\leqslant\boldsymbol{\alpha}} \to \mathcal{X}_{\leqslant\boldsymbol{\alpha}}$ is smooth, surjective, affine, and has fibers are isomorphic to $\mathrm{Res}_{k_{F_{\mathrm{cyc}}}/\mathbb{F}_p} U^{\check{M}} \rtimes \check{T}$.

Moreover, the formation of $X_{\leqslant\boldsymbol{\alpha}}$ tautological and does not depend on the choice of the total ordering on $\Phi_{\mathbb{K}}$.

Proof. Similar to Proposition 3.4. $\square$

12.3. **Twisted Weil-Deligne stacks.** For each relative root $\boldsymbol{\alpha} \in \Phi^+$, set $\mathbb{G}_{m,\boldsymbol{\alpha}} := \prod_{\alpha\in\boldsymbol{\alpha}} \mathbb{G}_{m,\alpha}$. Let

$$\mathcal{W}_{U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle} \to \mathcal{X}_{U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}$$

be the morphism relatively representing the presheaf

$$R \mapsto \mathrm{Tor}_1^{\mathcal{O}_{X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}}}(H^2_{\mathrm{Herr}}(\mathbf{E}_{F,\mathcal{O}_{X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}}}), R).$$

Note that

$$\mathrm{Tor}_1^{\mathcal{O}_{X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}}}(H^2_{\mathrm{Herr}}(\mathbf{E}_{F,\mathcal{O}_{X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}}}), R) = \mathrm{Tor}_1^{\mathcal{O}_{X_{\mathbb{G}_{m,\boldsymbol{\alpha}}}}}(H^2_{\mathrm{Herr}}(\mathbf{E}_{F_n,\mathcal{O}_{X_{\mathbb{G}_{m,\boldsymbol{\alpha}}}}}), R)^{\mathrm{Gal}(F_n/F)}. \tag{8}$$

Lemma 12.3. *We have*

$$\mathcal{W}_{U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle} = (\mathcal{W}_{F_n,U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}} \times_{\mathcal{X}_{F_n,\mathbb{G}_{m,\boldsymbol{\alpha}}}} \mathcal{X}_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle})^{\mathrm{Gal}(F_n/F)}.$$

Proof. Follows from Eq. (8) $\square$

Set

$$W_{U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle} := X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle} \times_{\mathcal{X}_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}} \mathcal{W}_{U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}$$

for the destackified Weil-Deligne stack. We add a subscript F to W or X when the ground field must be emphasized. Note that each connected component of

$$W_{F_n,U_{\boldsymbol{\alpha}}\rtimes\mathbb{G}_{m,\boldsymbol{\alpha}}} \times_{X_{F_n,\mathbb{G}_{m,\boldsymbol{\alpha}}}} X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}$$

is of the form $\mathrm{Spec}\,\bar{\mathbb{F}}_p[t_1^{\pm 1},\cdots,t_k^{\pm 1},s_1,\cdots,s_{k'}]$; the $\mathrm{Gal}(F_n/F)$-action fixes the t_*-parameters and permutes the s_*-parameters.

Definition 12.4. A connected component of $W_{U_{\boldsymbol{\alpha}} \rtimes \mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}$ is either of the form

$$\mathbb{G}_{m,\boldsymbol{\alpha}}$$

or of the form

$$\mathbb{G}_{m,\boldsymbol{\alpha}}[s]/(s\ u)$$

where $u \in \mathcal{O}_{\mathbb{G}_{m,\boldsymbol{\alpha}}}$ is the equation defining the closed subscheme of $\mathbb{G}_{m,\boldsymbol{\alpha}}$ that acts on $U_{\boldsymbol{\alpha}}$ via the cyclotomic character $\bar{\mathbb{F}}_p$-pointwise when restricted to Gal_{F_n}.

We call $\mathbb{G}_{m,\boldsymbol{\alpha}}/(u) \times \operatorname{Spec} \bar{\mathbb{F}}_p[s] \subset \mathbb{G}_{m,\boldsymbol{\alpha}}[s]/(s\ u)$ the Steinberg component.

12.4. **The b_{std} and c_{std} cochains.** Let $\alpha \in \boldsymbol{\alpha}$ be any absolute root. Write F_d/F for the splitting field of α; so $d = \#\boldsymbol{\alpha}$.

Let $\operatorname{Spec} \bar{\mathbb{F}}_p[t_\alpha^{\pm 1}, s_\alpha]/(s_\alpha(t_\alpha - 1)) \subset W_{F_n, U_{\alpha^\vee} \rtimes \mathbb{G}_{m,\alpha}}$ be the connected component containing the Steinberg component. Set $R := \bigotimes_{\alpha \in \boldsymbol{\alpha}} \bar{\mathbb{F}}_p[t_\alpha^{\pm 1}, s_\alpha]/(s_\alpha(t_\alpha - 1))$.

Recall that there exists a cochain

$$c'_{\alpha,\mathrm{std}} \in C^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F_n,R}))$$

such that

$$d^1_{\mathrm{Herr}}(c'_{\alpha,\mathrm{std}}) = (t_\alpha - 1) b'_{\alpha,\mathrm{std}}$$

and that $[b'_{\alpha,\mathrm{std}}]$ generates $H^2_{\mathrm{Herr}}(U_{\alpha^\vee}(\mathbf{E}_{F_n,R}))$. Note that

$$b_{\boldsymbol{\alpha},\mathrm{std}} := \sum_{\sigma^j \in \mathrm{Gal}(F_n/F)} \sigma^j b'_{\alpha,\mathrm{std}}$$

generates $H^2_{\mathrm{Herr}}(U_{\boldsymbol{\alpha}}(\mathbf{E}_{F,R}))$, by Frobenius reciprocity. The α-coordinate of $b_{\boldsymbol{\alpha},\mathrm{std}}$ is

$$b_{\alpha,\mathrm{std}} := \sum_{\sigma^{jd} \in \mathrm{Gal}(F_n/F_d)} \sigma^{jd} b'_{\alpha,\mathrm{std}} \in C^2_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F_d,R})),$$

which is a coboundary if and only if $H^2_{\mathrm{Herr}}(U_{\boldsymbol{\alpha}}(\mathbf{E}_{F,R})) = 0$. Set $c_{\boldsymbol{\alpha},\mathrm{std}} := \sum_{\sigma^j \in \mathrm{Gal}(F_n/F)} \sigma^j c'_{\alpha,\mathrm{std}}$.

12.5. **Gauge cochains.** Set

$$W_{\mathbb{K} \rtimes \check{T}\langle w\rangle} := \prod_{\boldsymbol{\alpha} \in \boldsymbol{\Phi}} W_{U_{\boldsymbol{\alpha}} \rtimes \mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle} \underset{\prod_{\boldsymbol{\alpha} \in \boldsymbol{\Phi}} X_{\mathbb{G}_{m,\boldsymbol{\alpha}}\langle w\rangle}}{\times} X_{\check{T}\langle w\rangle}.$$

Let $C \subset W_{\mathbb{K} \rtimes \check{T}\langle w\rangle}$ be an irreducible component, and set $R = \mathcal{O}_C$. Write

$$\begin{aligned}
\Psi_0(C) &= \{\boldsymbol{\alpha} \in \boldsymbol{\Phi}^+ | H^0_{\mathrm{Herr}}(U_{\boldsymbol{\alpha}}(\mathbf{E}_{F,R})) \text{ is finite free of rank } 1\}\\
\Psi_2(C) &= \{\boldsymbol{\alpha} \in \boldsymbol{\Phi}^+ | H^2_{\mathrm{Herr}}(U_{\boldsymbol{\alpha}}(\mathbf{E}_{F,R})) \text{ is finite free of rank } 1\}\\
\overline{\Psi}_0(C) &= \{\boldsymbol{\alpha} \in \boldsymbol{\Phi}^+ | H^0_{\mathrm{Herr}}(U_{\boldsymbol{\alpha}}(\mathbf{E}_{F,R})) \neq 0\}\\
\overline{\Psi}_2(C) &= \{\boldsymbol{\alpha} \in \boldsymbol{\Phi}^+ | H^2_{\mathrm{Herr}}(U_{\boldsymbol{\alpha}}(\mathbf{E}_{F,R})) \neq 0\}.
\end{aligned}$$

We fix a representative $\alpha \in \boldsymbol{\alpha}$ once for all. For each α, we choose a set of cocycles

$$\{c'_{\alpha,1}, \cdots, c'_{\alpha,n[F:\mathbb{Q}_p]+n[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]}\} \subset Z^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F_n,R}))$$

which is stable under the semilinear $\mathrm{Gal}(F_n/F)$-action and forms a basis of $H^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F_n,R}))$. Set

$$c'_{w^j \cdot \alpha, i} := \sigma^j c'_{\alpha,i}$$

for $1 \leqslant i \leqslant n([F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p])$ and $0 \leqslant j < \#\alpha =: d$. Also write

$$\sigma^{jd} c'_{\alpha,i} = c'_{\alpha,\sigma^{jd}(i)}.$$

Definition 12.5. We recursively define finitely many 1-cochains $\mathfrak{C}_{\Xi_{\alpha,k}} = \{c_{\alpha,\xi}\} \subset C^1_{\mathrm{Herr},+}(U_{\alpha}(\mathbf{E}_{F,R}))$, $\xi \in \Xi_{\alpha,k}$ for $k = 1,\dots,h_{\breve{G}}$.

We set

$$\Xi_{\alpha,1} = \begin{cases} \{1,2,\dots,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p],\mathrm{std}\} & \alpha \in \overline{\Psi}_2(C) \\ \{1,2,\dots,[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]\} & \text{otherwise} \end{cases}$$

and $\mathfrak{C}_{\Xi_{\alpha,1}} = \{c_{\alpha,\xi} | \xi \in \Xi_{\alpha,1}\}$, where

$$c_{\alpha,\xi} = \sum_{\sigma^j \in \mathrm{Gal}(F_n/F)} \sigma^j c'_{\alpha,\xi}.$$

Suppose $\mathfrak{C}_{\Xi_{\alpha,k}}$ is already defined. For each sequence of roots $\beta_1+\cdots+\beta_s \supset \alpha$ and elements $c_{\beta_i,\xi_i} \in C_{\Xi_{\beta_i,k}}$, denote by

$$[c_{\beta_1,\xi_1},\dots,c_{\beta_s,\xi_s}]_{\alpha}$$

the α-factor of

$$[c_{\beta_1,\xi_1},\dots,c_{\beta_s,\xi_s}] \in C^1_{\mathrm{Herr},+}((\prod_{\beta\in\beta_1+\cdots+\beta_s} U_{\beta^\vee})(\mathbf{E}_{F,R})).$$

We choose an element $c_{\alpha,[\xi_1,\xi_2,\dots,\xi_s]} \in C^1_{\mathrm{Herr},+}(U_{\alpha^\vee}(\mathbf{E}_{F,R}))$ such that

$$[c_{\beta_1,\xi_1},\dots,c_{\beta_s,\xi_s}]_{\alpha} = \begin{cases} a_{\alpha,[\xi_1,\dots,\xi_s]} b_{\alpha,\mathrm{std}} - d^1_{\mathrm{Herr}}(c_{\alpha,[\xi_1,\xi_2,\dots,\xi_s]}) & \alpha \in \overline{\Psi}_2(C) \\ -d^1_{\mathrm{Herr}}(c_{\alpha,[\xi_1,\xi_2,\dots,\xi_s]}) & \text{otherwise} \end{cases}$$

where $a_{\alpha,*} \in R$ is the unique scalar that makes the equation hold. We set

$$\Xi_{\alpha,k+1} = \Xi_{\alpha,k} \cup \{[\xi_1,\dots,\xi_k], \xi_i \in \mathfrak{C}_{\Xi_{\beta_i,k}}\}$$

and

$$\mathfrak{C}_{\Xi_{\alpha,k+1}} = \{c_{\alpha,\xi} : \xi \in \Xi_{\alpha,k+1}\}.$$

We call elements of $\mathfrak{C}_{\Xi_{*,*}}$ the *gauge cochains*.

Lemma 12.6. *If all irreducible components of $X_{<\alpha}(C)$ have dimension $\geqslant d$, then all irreducible components of $X_{\leqslant\alpha}(C)$ have irreducible components of dimension $\geqslant d+([F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p])\#\alpha$.*

Proof. Completely similar to Corollary 6.8. □

12.6. **The fine stratification.**

Definition 12.7. For subsets $\Psi_0,\Psi_2 \subset \Phi^+$ such that Ψ_2 is *not* non-obstructing, denote by

$$X_{\mathbb{K}\rtimes\check{T}\langle w\rangle}\langle\Psi\rangle \subset X_{\mathbb{K}\rtimes\check{T}\langle w\rangle}$$

the locally closed subscheme where $H^i_{\mathrm{Herr}}(U_\alpha)$ is finite free of rank 1 if and only if $\alpha \in \Psi_i$, for $i \in \{0,2\}$.

Theorem 12.8. (1) $X_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle$ is a (possibly empty) union of certain connected components of

$$\operatorname{Spec}\bar{\mathbb{F}}_p\begin{bmatrix} x_{\boldsymbol{\alpha},1},\ldots,x_{\boldsymbol{\alpha},[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p],s_{\boldsymbol{\alpha}}} & : & \boldsymbol{\alpha}\in\Phi_{\mathbb{K}} \\ t_\delta^{\pm 1} & : & \delta\in\Delta\end{bmatrix}/I$$

where I is generated by

$$\left\{\begin{matrix} s_{\boldsymbol{\alpha}} & : & \boldsymbol{\alpha}\notin\Psi_2 \\ r_{\boldsymbol{\alpha}} & : & \boldsymbol{\alpha}\in\Psi_2,\mathrm{ht}_{\mathbb{K}}(\boldsymbol{\alpha})>1 \\ (t_\alpha-1)^n & : & \alpha\in\bigcup_{\boldsymbol{\alpha}\in\Psi_0\cup\Psi_2}\boldsymbol{\alpha} \\ x_{\boldsymbol{\alpha},i}=\sigma^{j\#\boldsymbol{\alpha}}x_{\boldsymbol{\alpha},i} & : & 0\leqslant j\#\boldsymbol{\alpha}<n, 1\leqslant i\leqslant[F:\mathbb{Q}_p]+[k_{F_{\mathrm{cyc}}}:\mathbb{F}_p]\end{matrix}\right\}.$$

Here,

$$r_{\boldsymbol{\alpha}}:=\sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_m=\boldsymbol{\alpha},\xi_1,\ldots,\xi_m}\frac{1}{m!}a_{\boldsymbol{\alpha},[\xi_1,\ldots,\xi_m]}\prod_{j=1}^m\mu_{\boldsymbol{\beta}_j,\xi_j}$$

where $\mu_{\boldsymbol{\beta}_j,\xi_j}$ is the coefficient before $c_{\boldsymbol{\beta}_j,\xi_j}$ in the universal cochain $c_{\boldsymbol{\beta}_j}^{\mathrm{univ}}$.

(2) We can normalize it so that

$$r_{\boldsymbol{\alpha}}=\sum_{\beta\in\boldsymbol{\beta}\in\Phi_{\mathbb{K}},\delta\in\boldsymbol{\delta}\in\Delta_{\mathbb{K}},\delta+\beta\supset\boldsymbol{\alpha}}\frac{1}{2}N_{\beta,\delta}x_{\boldsymbol{\beta},1}x_{\boldsymbol{\delta},1}+Q_{\boldsymbol{\alpha}}$$

where $Q_{\boldsymbol{\alpha}}$ does not depend on $x_{\boldsymbol{\beta},1}$ for $\mathrm{ht}_{\mathbb{K}}(\boldsymbol{\beta})=\mathrm{ht}_{\mathbb{K}}(\boldsymbol{\alpha})-1$.

Proof. Completely similar to Theorem 7.6. □

Definition 12.9. Write $f^{\mathrm{univ}},g^{\mathrm{univ}}\in(\mathbb{K}\rtimes\breve{T}\langle w\rangle)(\mathbf{E}_{F,\mathcal{O}_{X_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle}})$ for the universal étale (φ,Γ)-module satisfying

$$f^{\mathrm{univ}}\varphi(g^{\mathrm{univ}})=g^{\mathrm{univ}}\gamma(f^{\mathrm{univ}}).$$

Set

$$\log_{\leqslant h_{\check{G}}}(f^{\mathrm{univ}}(f^{\mathrm{univ}})^{\mathrm{ss}-1},g^{\mathrm{univ}}(g^{\mathrm{univ}})^{\mathrm{ss}-1})=\sum_{\boldsymbol{\alpha}\in\Phi_{\mathbb{K}}}c_{\boldsymbol{\alpha}}$$

Denote by

$$\dot{X}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\subset X_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}$$

the quasi-affine open subscheme where $[c_{\boldsymbol{\delta}}]\neq 0\in H^1_{\mathrm{Herr}}(U_{\boldsymbol{\delta}})$ for $\boldsymbol{\delta}\in\Delta_{\mathbb{K}}$.

Corollary 12.10. (1) If the Jacobian

$$J=(\frac{\partial r_{\boldsymbol{\alpha}}}{\partial x_{\boldsymbol{\beta},1}})$$

is of full rank, then $\dot{X}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle$ is smooth. Write $\dot{W}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle\subset W_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}$ for the image of $\dot{X}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle$, then all fibers of $\dot{X}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle\to\dot{W}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle$ are irreducible of dimension $[F:\mathbb{Q}_p]\#\bigcup_{\boldsymbol{\alpha}\in\Phi_{\mathbb{K}}}\boldsymbol{\alpha}$.

(2) Write r for the corank of J. Then all fibers of $\dot{X}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle\to\dot{W}_{\mathbb{K}\rtimes\breve{T}\langle w\rangle}\langle\Psi\rangle$ are of dimension at most $[F:\mathbb{Q}_p]\#\bigcup_{\boldsymbol{\alpha}\in\Phi_{\mathbb{K}}}\boldsymbol{\alpha}+r$.

Proof. Completely similar to Corollary 9.11. □

It turns out that for the twisted situation, we don't even need the full power of the cone model.

Theorem 12.11. Let $C \subset \mathcal{W}_{U^{\check{M}} \rtimes \check{T}\langle w\rangle}$ be an irreducible component.

(1) $\mathcal{W}_{U^{\check{M}} \rtimes \check{T}\langle w\rangle}$ is equidimensional of dimension 0.

(2) If $[F : \mathbb{Q}_p] \geqslant \#\Phi^+ + 2$, then each $\mathcal{X}_{U^{\check{M}} \rtimes \check{T}\langle w\rangle}(C)$ is irreducible. As a consequence, $\mathcal{X}_{U^{\check{M}} \rtimes \check{T}\langle w\rangle} \to \mathcal{W}_{U^{\check{M}} \rtimes \check{T}\langle w\rangle}$ is generically smooth and induces a bijection of irreducible components.

(3) If $\check{G} = \mathrm{F}_4$, then part (2) holds without the $[F : \mathbb{Q}_p] \geqslant \#\Phi^+ + 2$ assumption.

Proof. (1) Let

$$S_C := \Psi_2(C) \cap \Delta$$

be the relative height-one orbits on which C is Steinberg. A relative height-one root has coefficient one in a unique simple-root direction outside $\Delta_{\check{M}}$ and coefficient zero in every other omitted direction. Since $w \in W_{\check{M}}$, this direction is constant on each cyclic orbit, giving a map

$$\pi : S_C \longrightarrow \Delta - \Delta_{\check{M}}.$$

We only need this map on S_C, not a bijection on all of Δ.

The map π is injective. Indeed, argue contrapositively. In a fixed omitted direction, the relative-height-one slice is connected by Levi root strings. The root-string decomposition under the cyclic action has the following dichotomy: either one cyclic orbit contains adjacent roots whose difference is in $\Phi_{\check{M}}$, or two distinct cyclic orbits in that direction contain such an adjacent pair. Thus two elements of S_C with the same image under π would make S_C non-obstructing. This is impossible because $S_C \subset \Psi_2(C)$ and the fine stratum requires $\Psi_2(C)$ to be not non-obstructing.

Each $\delta \in S_C$ contributes one Steinberg additive parameter, so the additive dimension gain is $\#S_C$. On the multiplicative side, all relative roots in one omitted direction impose one torus condition after quotienting by the characters $\delta - w\delta$. Conditions in distinct omitted directions are independent, as their images are distinct fundamental-coweight coordinates modulo the Levi character lattice. Hence the multiplicative dimension loss is $\#\pi(S_C)$. Injectivity gives

$$\#S_C = \#\pi(S_C),$$

so the additive gain exactly cancels the multiplicative loss. More additive gains would force two elements over one omitted direction and hence violate the non-obstructing constraint. Therefore every allowed component has dimension zero. Over F_n, the restricted actions on $U_{w\delta^\vee}$ and $U_{\delta^\vee}$ need not be trivial but are equal, so the indicated torus quotient is compatible with the root-group factors.

(2) Follows from Corollary 12.10 and Theorem 10.11.

(3) By Part (1), it remains to show when $\mathbb{K} \neq U^{\check{M}}$, the corank of $J = (\frac{\partial r_{\boldsymbol{\alpha}}}{\partial x_{\boldsymbol{\beta},1}})$ does not exceed

$$\# \bigcup_{\alpha \in \Phi^+ - \Phi_{\mathbb{K}}} \alpha - \dim \mathcal{W}_{\mathbb{K} \rtimes \check{T}\langle w\rangle}.$$

We also have

$$\dim \mathcal{W}_{\mathbb{K} \rtimes \check{T}\langle w\rangle} \leqslant \#\Psi_2 \cap \Delta_{\mathbb{K}} - \dim \frac{\operatorname{span}_{\mathbb{Q}}\{\alpha | \alpha \in \Psi_2 \cup \Psi_0\}}{\operatorname{span}_{\mathbb{Q}}\{\delta - w\delta | \delta \in \Delta\}}.$$

Since every term is easily computable by linear algebra, Part (3) follows from enumeration of all configurations of $(\mathbb{K}, \Psi)$. □

We remark that part (1) of Theorem 12.11 does not have to be true for general $\mathbb{K}$: in general, $\Delta_{\mathbb{K}}$ may have more elements than $\Delta - \Delta_{\check{M}}$.

Part III: The p-adic formal Weil-Deligne stacks

13. Geometrization of the p-adic local Euler characteristic

We use the integral and rigid moduli theory of [Lin25, Lemma 11 and Theorems 10–12]. For the twisted version, sections are additionally required to be Galois-stable. There is no other difference in the construction.

Lemma 13.1. Let $X = \operatorname{Spf} R$ where $R = \mathbb{Z}_p\langle T\rangle$. The presheaf over X defined by $S \mapsto \operatorname{Tor}_1^R(R/(T), S)$ is representable by the formal scheme $\operatorname{Spf}(R\langle Y\rangle/(TY))$.

Proof. First, compute the functor $F(S) = \operatorname{Tor}_1^R(R/(T), S)$. Since T is a non-zero-divisor in R, we have a free resolution:

$$0 \to R \xrightarrow{\cdot T} R \to R/(T) \to 0$$

Tensoring with an R-algebra S, we get the complex:

$$0 \to S \xrightarrow{\cdot T} S \to S/TS \to 0$$

The first homology group is $\ker(S \xrightarrow{\cdot T} S) = S[T]$. Thus, $F(S)$ assigns each S its T-torsion submodule, $S[T]$.

To represent this functor, consider the p-adically complete R-algebra $A = R\langle Y\rangle/(TY)$. A continuous R-algebra homomorphism $\phi : A \to S$ is determined by $y = \phi(Y) \in S$. This requires $Ty = 0$ in S. Because A is a Tate algebra, any such y gives a continuous map. This yields a bijection $\operatorname{Hom}_R(A, S) \cong S[T]$. □

Consider the moduli stack $\mathcal{X}^{0,\mathrm{tame}}_{F,\mathbb{G}_m}$ of potentially semistable rank 1 étale (φ, Γ)-module of trivial Hodge type and tame inertial type over $\mathbf{A}_{F,\bar{\mathbb{Z}}_p}$. Here, $\mathbf{A}_{F,A}$ is the usual thickening of $\mathbf{E}_{F,A}$. All such étale (φ, Γ)-modules correspond to unramified twists of Teichmüller lifts of a mod p Galois representation. So, we have

$$\mathcal{X}^{0,\mathrm{tame}}_{F,\mathbb{G}_m} = \coprod_{\tau\in\mathbb{F}_q^\times} \mathcal{X}^{0,\tau}_{F,\mathbb{G}_m} = \coprod_{\tau\in\mathbb{F}_q^\times} [\operatorname{Spf} \bar{\mathbb{Z}}_p\langle T^{\pm1}\rangle / \operatorname{Spf} \bar{\mathbb{Z}}_p\langle T^{\pm1}\rangle],$$

and admits a destackification

$$X^{0,\mathrm{tame}}_{F,\mathbb{G}_m} := \coprod_{\tau\in\mathbb{F}_q^\times} \operatorname{Spf} \bar{\mathbb{Z}}_p\langle T^{\pm1}\rangle.$$

Write $M^{0,\mathrm{tame}}$ for the universal étale (φ, Γ)-module over $R = \mathbf{A}_{F,\bar{\mathbb{Z}}_p\langle T^{\pm1}\rangle}$. Write M_{cyc} for the cyclotomic étale (φ, Γ)-module over $\mathbf{A}_{F,\bar{\mathbb{Z}}_p}$. Consider the presheaf

$$S \mapsto \operatorname{Tor}_1^R(H^2_{\mathrm{Herr}}(M^{0,\mathrm{tame}} \otimes M_{\mathrm{cyc}}), S) = \operatorname{Tor}_1^R(\bar{\mathbb{Z}}_p\langle T^{\pm1}\rangle/(T-1), S),$$

which is representable by

$$\operatorname{Spf} \bar{\mathbb{Z}}_p\langle T^{\pm1}, Y\rangle/(Y(T-1)).$$

The formal scheme $\operatorname{Spf} \bar{\mathbb{Z}}_p\langle T^{\pm1}, Y\rangle/(Y(T-1))$ has two irreducible components

$$\operatorname{Spf} \bar{\mathbb{Z}}_p\langle T^{\pm1}\rangle \cup \operatorname{Spf} \bar{\mathbb{Z}}_p\langle Y\rangle$$

with special fiber

$$\operatorname{Spec} \bar{\mathbb{F}}_p[T^{\pm1}] \cup \operatorname{Spec} \bar{\mathbb{F}}_p[Y]$$

and generic fiber

$$\operatorname{Sp} \bar{\mathbb{Q}}_p\langle T^{\pm 1}\rangle \cup \operatorname{Sp} \bar{\mathbb{Q}}_p\langle Y\rangle.$$

Lemma 13.2. If $\breve{G}$ has connected center, then there exists a cocharacter $\lambda \in X_*(\breve{T})$ such that

$$\langle \lambda, \delta\rangle = -1$$

for all $\delta \in \Delta$.

Proof. Recall that "having simply-connected derived subgroup" $\Leftrightarrow$ "having fundamental weights". Dualizing both sides, we get "having connected center" $\Leftrightarrow$ "having fundamental coweights". Take λ to be the negative sum of fundamental coweights. $\square$

We fix such a cocharacter λ once for all. Let $w \in N_{\breve{G}}(\breve{T})/\breve{T}$ be a Weyl group element of order n, that is elliptic in a proper Levi $\breve{M} \subset \breve{G}$. Let

$$\chi_{\mathrm{LT}} : \mathrm{Gal}_{F_n} \to \mathbb{G}_m(\mathcal{O}_{F_n}) = \mathcal{O}_{F_n}^{\times}$$

be the Lubin-Tate character. Write $\chi_{\mathrm{LT}}^{\lambda}$ for the composite

$$\mathrm{Gal}_{F_n} \xrightarrow{\chi_{\mathrm{LT}}} \mathbb{G}_m(\mathcal{O}_{F_n}) \xrightarrow{\lambda(\mathcal{O}_{F_n})} \breve{T}(\mathcal{O}_{F_n}).$$

Fix a lift $\sigma \in \mathrm{Gal}_F$ of a generator of $\mathrm{Gal}(F_n/F)$. We define the Galois representation $N_w(\chi_{\mathrm{LT}}^{\lambda}) : \mathrm{Gal}_F \to \breve{T}(\bar{\mathbb{Z}}_p)\langle w\rangle$ by setting:

$$N_w(\chi_{\mathrm{LT}}^{\lambda})(g\sigma^k) := \left(\prod_{i=0}^{n-1} w^i \chi_{\mathrm{LT}}^{\lambda}(\sigma^{-i} g \sigma^i) w^{-i}\right) w^k.$$

Lemma 13.3. *Let $\delta^\vee \in \Delta - \Delta_M$ and set $\boldsymbol{\delta} := \langle w\rangle \cdot \delta^\vee$. we have*

$$H^2(\mathrm{Gal}_F, U_{\boldsymbol{\delta}}(\bar{\mathbb{Z}}_p)) \cong \bar{\mathbb{Z}}_p.$$

Proof. Write F_m for the splitting field of δ. By Shapiro's Lemma, we have an isomorphism of cohomology groups:

$$H^2(\mathrm{Gal}_F, U_{\boldsymbol{\delta}}(\bar{\mathbb{Z}}_p)) \cong H^2(\mathrm{Gal}_{F_m}, U_{\delta^\vee}(\bar{\mathbb{Z}}_p)) = H^2(\mathrm{Gal}_{F_m}, \bar{\mathbb{Z}}_p(1)) = \bar{\mathbb{Z}}_p. \qquad \square$$

Definition 13.4. Write $\underline{\lambda}_w \in X^*(\mathrm{Res}_{F/\mathbb{Q}_p} \breve{T})$ for the Hodge type of $N_w(\chi_{\mathrm{LT}}^{\lambda})$.

Write

$$\mathcal{X}_{U^{\breve{M}} \rtimes \breve{T}\langle w\rangle}^{\underline{\lambda}_w, \mathrm{tame}}$$

for the moduli stack of potentially semistable Galois representations of Hodge type $\underline{\lambda}_w$ and tame inertial type. Write

$$\mathfrak{X}_{U^{\breve{M}} \rtimes \breve{T}\langle w\rangle}^{\underline{\lambda}_w, \mathrm{tame}}$$

for the rigid generic fiber of $\mathcal{X}_{U^{\breve{M}} \rtimes \breve{T}\langle w\rangle}^{\underline{\lambda}_w, \mathrm{tame}}$.

Next, we construct the integral Weil-Deligne stack.

Definition 13.5. Let $R = \mathcal{O}_{X^{\underline{\lambda}_w,\mathrm{tame}}_{\check{T}\langle w\rangle}}$. Define $\mathscr{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle} \to \mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{\check{T}\langle w\rangle}$ to be the morphism that represents the presheaf

$$S \mapsto \prod_{\delta\in\Delta} \mathrm{Tor}_1^R(H^2_{\mathrm{Herr}}(U_\delta(\mathbf{A}_{F,R})), S).$$

Write $\mathfrak{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ for the rigid generic fiber of $\mathscr{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$, whose representability follows from unramified descent of prime-to-p degree. Note that the special fiber of $\mathscr{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ is precisely $\mathcal{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$.

We recall the following theorem from our previous work:

Theorem 13.6. *([Lin25, Theorem 3])* $\mathfrak{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle} \to \mathfrak{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ *induces a bijection of irreducible components.*

Proof. This is used as the external input [Lin25, Theorem 3]. □

Corollary 13.7. The special fiber of $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ is equidimensional of dimension $[F:\mathbb{Q}_p]\#\Phi^{\check{M}}$, and its number of irreducible components is at least as many as the number of irreducible components of $\mathcal{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$.

Proof. Note that irreducible components of $\mathcal{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ and $\mathfrak{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ are precisely the special fiber and the rigid generic fiber of irreducible components $\mathscr{C}$ of the p-adic formal stack $\mathscr{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$.

Consider

$$\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}(\mathscr{C}) := \mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle} \times_{\mathscr{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}} \mathscr{C},$$

whose rigid generic fiber $\mathfrak{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}(\mathscr{C})$ is of dimension $[F:\mathbb{Q}_p]\#\Phi^{\check{M}}$ and non-empty by Theorem 13.6. Since $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}(\mathscr{C})$ is a closed sub-formal scheme of $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$, the Zariski closure $\mathcal{Y}$ of $\mathfrak{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}(\mathscr{C})$ in $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ is contained in $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}(\mathscr{C})$. Take $\mathcal{Y}$ to be the schematic closure of the indicated nonempty rigid component, with its p-power torsion removed. It is then $\bar{\mathbb{Z}}_p$-flat by construction. A nonzero flat, topologically finite-type formal scheme over the complete DVR (all stacks descend to a DVR) cannot have empty special fiber: otherwise p would be a unit on its coordinate ring, contrary to adic completeness. Hence $\mathcal{Y}\otimes\bar{\mathbb{F}}_p$ is nonempty. Flatness and Krull's Hauptidealsatz give $\dim(\mathcal{Y}\otimes\bar{\mathbb{F}}_p) = \dim\mathcal{Y} - 1 = [F:\mathbb{Q}_p]\#\Phi^{\check{M}}$. This argument is applied separately to every $\mathscr{C}$. So, there exists at least one irreducible component of $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}\otimes\bar{\mathbb{F}}_p$ that maps onto $C := \mathscr{C}\otimes\bar{\mathbb{F}}_p$ and does not map onto other irreducible components of $\mathcal{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$.

The equidimensionality follows from the Krull's Hauptidealsatz argument applied to the entire stack $\mathcal{X}^{\underline{\lambda}_w,\mathrm{tame}}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$. □

Theorem 13.8. If the number of irreducible components of $\mathcal{X}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ does not exceed the number of irreducible components of $\mathcal{W}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$, then all $\bar{\mathbb{F}}_p$-points of $\mathcal{X}_{U^{\check{M}}\rtimes\check{T}\langle w\rangle}$ have a potentially semistable lift of regular Hodge type $\underline{\lambda}_w$ and tame inertial type.

The condition is satisfied in the following cases:

- $\check{G}$ and w arbitrary and $[F:\mathbb{Q}_p] \geqslant \#\Phi^+ + 2$,
- $\check{G} = \mathrm{F}_4$ and $w \neq 1$, or

- $\breve{G} = \mathrm{F}_4$ and $F \neq \mathbb{Q}_p$.

Proof. Combine Theorem 12.11, Theorem 9.13, Theorem 8.1 and Corollary 13.7. □

14. The qualitative Breuil-Mézard conjecture

Lemma 14.1. The scheme-theoretic image Y of each

$$\dot{\mathcal{X}}_{\breve{B}}\langle\Psi\rangle \to \mathcal{X}_{\breve{G}}^{\mathrm{EG}}$$

has dimension at least $\dim \dot{\mathcal{X}}_{\breve{B}}\langle\Psi\rangle$.

Proof. It suffices to show for each $\operatorname{Spec} \bar{\mathbb{F}}_p \to Y$, the fiber

$$\operatorname{Spec} \bar{\mathbb{F}}_p \times_Y \dot{\mathcal{X}}_{\breve{B}}\langle\Psi\rangle$$

has dimension (at most) 0. Suppose this $\bar{\mathbb{F}}_p$-point corresponds to a Galois representation $\bar{\rho} : \mathrm{Gal}_F \to \breve{B}(\bar{\mathbb{F}}_p)$, then this fiber embed in $\mathrm{Aut}_{\breve{G}}(\bar{\rho})/\mathrm{Aut}_{\breve{B}}(\bar{\rho}) = \{1\}$, as $\bar{\rho}$ is maximally non-split:

Sublemma 14.1.1. *Let $g = tu \in \breve{B}$ where $t \in \breve{T}$ and $u \in U$, such that $\log_{\leqslant h_{\breve{G}}} u = \sum_{\alpha\in\Phi^+} u_\alpha$. If $u_\delta \neq 0$ for all simple roots $\delta \in \Delta$, then $Z_{\breve{G}}(u) \subset \breve{B}$.*

Proof. Let $z \in Z_{\breve{G}}(u)$. By the Bruhat decomposition, we can uniquely write $z = b\dot{w}v$, where:

- $b \in \breve{B}$,
- $\dot{w}$ is a representative of a Weyl group element,
- $v \in U_w = U \cap \dot{w}^{-1}U^-\dot{w} \subset U$.

Since z centralizes u, we have $z^{-1}uz = u$. Substituting the Bruhat decomposition of z, we get:

$$(b\dot{w}v)^{-1}u(b\dot{w}v) = u \implies \dot{w}^{-1}(b^{-1}ub)\dot{w} = vuv^{-1}$$

Let $u_1 = b^{-1}ub$. Because $b \in \breve{B}$, the conjugation action of b preserves U and its lower central series. Specifically, the induced action on the abelianization $U/[U,U] \cong \prod_{\delta\in\Delta} \mathfrak{g}_\delta$ is given by non-zero scalars on each simple root space. Therefore, u_1 is also regular unipotent, meaning $\log u_1$ has a non-zero component in $\mathfrak{g}_\delta$ for every $\delta \in \Delta$.

On the right side of the equation, let $u_2 = vuv^{-1}$. Since both v and u are in U, $u_2 \in U$.

We now have the equality $\dot{w}^{-1}u_1\dot{w} = u_2$. Passing to the Lie algebra, this gives:

$$\mathrm{Ad}(\dot{w}^{-1}) \log u_1 = \log u_2$$

Assume for the sake of contradiction that $\dot{w} \neq 1$. By the standard properties of the Weyl group, there exists at least one simple root $\delta \in \Delta$ such that $w^{-1}(\delta) < 0$ (i.e., it is a negative root).

Because u_1 is regular unipotent, $\log u_1$ has a non-zero component $X_\delta \in \mathfrak{g}_\delta$. Under the adjoint action of $\dot{w}^{-1}$, this component is mapped to $\mathrm{Ad}(\dot{w}^{-1})X_\delta$, which is a non-zero element in the negative root space $\mathfrak{g}_{w^{-1}(\delta)}$. This contradiction forces $\dot{w} = 1$. □

□

Proposition 14.2. If

- $\breve{G}$ is arbitrary and $[F : \mathbb{Q}_p] \geqslant \#\Phi^+ + 2$ or
- $\breve{G} = \mathrm{F}_4$ and $F \neq \mathbb{Q}_p$,

then $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ is equidimensional and its irreducible components are in bijection with the irreducible components of $\mathcal{W}_{\breve{B}}$.

Proof. By Theorem 13.8, $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ is contained in the special fiber of $\mathcal{X}^{\lambda_1,\mathrm{tame}}_{\breve{G}}$, which is equidimensional of dimension $[F:\mathbb{Q}_p]\#\Phi^+$. Here the crystalline stack is a closed substack of the integral Emerton–Gee stack by its scheme-theoretic-image construction. Conversely, Theorem 13.8 gives a closed immersion from the reduced Emerton–Gee stack into the reduced special fiber of this crystalline stack. The two opposite closed immersions identify these reduced substacks. Hence $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ is the reduced special fiber of the indicated crystalline stack and is equidimensional of the same dimension; in particular, there are no additional lower-dimensional irreducible components. By Lemma 14.1, the image of $\dot{\mathcal{X}}_{\breve{B}}(C)$ in $\mathcal{X}^{\mathrm{EG}}_{\breve{G}}$ is a top-dimensional irreducible component for each irreducible component $C\subset\mathcal{W}_{\breve{B}}$. We remark that $\dot{\mathcal{X}}_{\breve{B}}(C)$ is always irreducible, despite that $\mathcal{X}_{\breve{B}}(C)$ may not be. □

Lemma 14.3. The morphism

$$\mathcal{X}_{K\rtimes\breve{T}}[K]\langle\Psi\rangle\to\mathcal{X}^{\mathrm{EG}}_{\breve{G}}$$

factors through $[\mathcal{X}_{K\rtimes\breve{T}}[K]\langle\Psi\rangle/\prod_{\beta\in\Psi_0}U_{(-\beta)^\vee}]$.

So, its scheme-theoretic image has dimension $\leqslant\dim\mathcal{X}_{K\rtimes\breve{T}}[K]\langle\Psi\rangle-\#\Psi_0$.

Proof. Note that $\mathcal{X}_{K\rtimes\breve{T}}[K]\langle\Psi\rangle$ is clearly stable under the conjugation action of the negative root group action of $U_{(-\beta)^\vee}$ for $\beta\in\Psi_0$. □

Corollary 14.4. When $\breve{G}=\mathrm{F}_4$ and $F=\mathbb{Q}_p$, the top-dimensional irreducible components $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ are in bijection with the irreducible components of $\mathcal{W}_{\breve{B}}$.

The only possibilities for the non-top-dimensional irreducible components are the scheme-theoretic image of $\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle$ for the configurations (K,Ψ) listed in Table 1.

We don't know whether $\mathcal{X}^{\mathrm{EG}}_{\mathrm{F}_4,\mathrm{red}}$ is equidimensional yet.

Proof. Combine Lemma 14.3, Theorem 8.1, and the proof of Proposition 14.2. □

15. Rotation by negative root groups

In this section, we treat the 4 extra components of $\mathcal{X}_{\breve{B}}$ for $\breve{G}=\mathrm{F}_4$ and $F=\mathbb{Q}_p$.

ID	$\Phi^+-\Phi_K$	Ψ_0	Ψ_2
I	$\{0010\}$	$\{0010\}$	$\{0001,0011,0100,0110,0120\}$
II	$\{0010\}$	$\{0010\}$	$\{0001,0011,0100,0110,0120,1000\}$
III	$\{1000,0010\}$	$\{0010,1000\}$	$\left\{\begin{matrix}0001,0011,0100,0110\\0120,1100,1110,1120\end{matrix}\right\}$
IV	$\{1000,0001,0011,0010\}$	$\{0001,0010,0011,1000\}$	$\left\{\begin{matrix}0100,0110,0111,0120,0121,0122\\1100,1110,1111,1120,1121,1122\end{matrix}\right\}$

The goal is to prove the following:

Proposition 15.1. (Theorem 15.6) There exists a Zariski dense substack $\mathcal{U}\subset\mathcal{X}_{K\rtimes\breve{T}}\langle\Psi\rangle$ where (K,Ψ_0,Ψ_2) is one of the four configurations above, such that for each $\bar{\mathbb{F}}_p$-point of $\mathcal{U}$ corresponding to $\bar{\rho}:\mathrm{Gal}_F\to(K\rtimes\breve{T})(\bar{\mathbb{F}}_p)\subset\breve{B}(\bar{\mathbb{F}}_p)$, there exists

- a deformation
$$\rho : \mathrm{Gal}_F \to \breve{B}(\bar{\mathbb{F}}_p[\![\varepsilon]\!])$$
- an element $\mu \in \bar{\mathbb{F}}_p[\![\varepsilon]\!]$, and
- a root $\delta_{\mathrm{rot}} \in \Psi_0$

satisfying
$$\bar{\rho} = \exp(-\mu e_{-\delta_{\mathrm{rot}}})\rho\exp(\mu e_{-\delta_{\mathrm{rot}}}) \mod \varepsilon$$
and $\rho[\frac{1}{\varepsilon}]$ is a $\bar{\mathbb{F}}_p(\!(\varepsilon)\!)$-point of $\dot{\mathcal{X}}_{\breve{B}} = \mathcal{X}_{\breve{B}}[U]$.

Remark 15.2. The proposition is reduced to Theorem 15.6 which is proved towards the end of this section. The insight here is that, even though these $\mathcal{X}_{\breve{B}}[K]\langle\Psi\rangle$ are not in the closure of $\dot{\mathcal{X}}_{\breve{B}}$, their scheme-theoretic image in $\mathcal{X}_{\breve{G}}^{\mathrm{EG}}$ is in the closure of the scheme-theoretic image of $\dot{\mathcal{X}}_{\breve{B}}$ in $\mathcal{X}_{\breve{G}}^{\mathrm{EG}}$.

The key observation is that $\Psi_2 \subset \Delta_K$. Write
$$V_K := \prod_{\alpha\in(\Phi^+-\Phi_K)\cup\Delta_K} U_{\alpha^\vee},$$
so there is a short exact sequence
$$1 \to \bar{K}_1 \to V_K \to \frac{U}{K} \to 1$$

Lemma 15.3. Proposition 15.1 is equivalent to the same statement after replacing $K \rtimes \breve{T}$ and $\breve{B}$ by $\bar{K}_1 \rtimes \breve{T}$ and $V \rtimes \breve{T}$: There exists a Zariski dense substack $\mathcal{U} \subset \mathcal{X}_{\bar{K}_1\rtimes\breve{T}}\langle\Psi\rangle$ where (K, Ψ_0, Ψ_2) is one of the four configurations above, such that for each $\bar{\mathbb{F}}_p$-point of $\mathcal{U}$ corresponding to $\bar{\rho} : \mathrm{Gal}_F \to (\bar{K}_1 \rtimes \breve{T})(\bar{\mathbb{F}}_p) \subset (V \rtimes \breve{T})(\bar{\mathbb{F}}_p)$, there exists

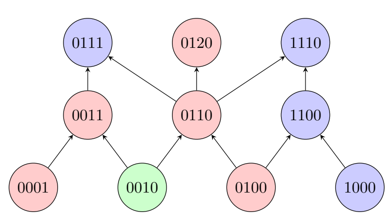
- a deformation
$$\rho : \mathrm{Gal}_F \to (V \rtimes \breve{T})(\bar{\mathbb{F}}_p[\![\varepsilon]\!])$$
- an element $\mu \in \bar{\mathbb{F}}_p[\![\varepsilon]\!]$, and
- a root $\delta_{\mathrm{rot}} \in \Psi_0$

satisfying
$$\bar{\rho} = \exp(-\mu e_{-\delta_{\mathrm{rot}}})\rho\exp(\mu e_{-\delta_{\mathrm{rot}}}) \mod \varepsilon$$
and that $\rho[\frac{1}{\varepsilon}]$ does not factor through any proper $\breve{T}$-stable normal subgroup of $V \rtimes \breve{T}$.

Proof. By Nakayama's lemma $H^2(\mathrm{Gal}_F, U_{\alpha^\vee}(\bar{\mathbb{F}}_p[\![\varepsilon]\!])) = H^2(\mathrm{Gal}_F, U_{\alpha^\vee}(\bar{\mathbb{F}}_p)) = 0$ for each $\alpha \in \Phi_K - \Delta_K$. □

It is clear that
$$X_{\bar{K}_1\rtimes\breve{T}}\langle\Psi\rangle = \mathrm{Spec}\,\bar{\mathbb{F}}_p[t_\delta^{\pm1}, x_{\alpha,1}, x_{\alpha,2}, s_\beta : \delta \in \Delta, \alpha \in \Delta_K, \beta \in \Psi_2]/(t_\alpha - 1 : \alpha \in \Psi_0 \cup \Psi_2).$$

Example 15.4. (Configuration I) We work out the first row of the table:



where Ψ_2 roots are colored by red and the Ψ_0 root is colored by green. It is harmless to assume $x_{\alpha,2} = 0$ as $\Psi_0 \cap \Delta_K = \varnothing \Rightarrow [c_{\alpha,2}] = 0 \in H^1(\mathrm{Gal}_F, U_{\alpha^\vee}(\bar{\mathbb{F}}_p))$; the variable $x_{\alpha,2}$ only plays the role of framing. We can also remove the roots $1000, 1100, 1110, 0111$ as they are not dominated by the Ψ_2 roots, and replace V by $\prod_{\alpha\in\{0001,0010,0100,0011,0110,0120\}} U_{\alpha^\vee}$. Write $x_\alpha := x_{\alpha,1}$.

The universal family of $X_{\bar{K}_1 \rtimes \check{T}}\langle\Psi\rangle$ is given by

$$\begin{aligned}
c^{\mathrm{univ}}_{0010} &= 0\\
c^{\mathrm{univ}}_{0001} &= x_{0001}c_{0001,1} + s_{0001}c_{0001,\mathrm{std}}\\
c^{\mathrm{univ}}_{0011} &= x_{0011}c_{0011,1} + s_{0011}c_{0011,\mathrm{std}}\\
c^{\mathrm{univ}}_{0100} &= x_{0100}c_{0100,1} + s_{0100}c_{0100,\mathrm{std}}\\
c^{\mathrm{univ}}_{0110} &= x_{0110}c_{0110,1} + s_{0110}c_{0110,\mathrm{std}}\\
c^{\mathrm{univ}}_{0120} &= x_{0120}c_{0120,1} + s_{0120}c_{0120,\mathrm{std}}.
\end{aligned}$$

Set $\delta_{\mathrm{rot}} := 0010$. Let μ be a free parameter. We replace $\bar{\rho}$ by

$$\bar{\rho}_\mu := \exp(\mu e_{-0010})\bar{\rho}\exp(-\mu e_{-0010})$$

where e_{-0010} is the standard Chevalley basis element. Note that $\bar{\rho}^{\mathrm{ss}}\exp(\mu e_{-0010}) = \exp(\mu e_{-0010})\bar{\rho}^{\mathrm{ss}}$. Also note that

$$\begin{aligned}
\exp(\mu e_{-0010})e_{0120}\exp(-\mu e_{-0010}) &= e_{0120} + \mu[e_{-0010}, e_{0120}] + \frac{\mu^2}{2}[e_{-0010}, [e_{-0010}, e_{0120}]]\\
&= e_{0120} - \mu e_{0110} - \mu^2 e_{0100}\\
\exp(\mu e_{-0010})e_{0110}\exp(-\mu e_{-0010}) &= e_{0110} + 2\mu e_{0100}\\
\exp(\mu e_{-0010})e_{0011}\exp(-\mu e_{-0010}) &= e_{0011} - \mu e_{0001}
\end{aligned}$$

This transforms the universal coordinates according to the rule

$$\begin{aligned}
c^{\mathrm{univ}}_{0110} &\mapsto c_{0110}(\mu) := c^{\mathrm{univ}}_{0110} - \mu c^{\mathrm{univ}}_{0120}\\
c^{\mathrm{univ}}_{0100} &\mapsto c_{0100}(\mu) := c^{\mathrm{univ}}_{0100} + 2\mu c^{\mathrm{univ}}_{0110} - \mu^2 c^{\mathrm{univ}}_{0120}\\
c^{\mathrm{univ}}_{0011} &\mapsto c_{0011}(\mu) := c^{\mathrm{univ}}_{0011}\\
c^{\mathrm{univ}}_{0001} &\mapsto c_{0001}(\mu) := c^{\mathrm{univ}}_{0001} - \mu c^{\mathrm{univ}}_{0011}.
\end{aligned}$$

Next, we construct the universal ε-deformation

$$\begin{aligned}
c_{0010}(\varepsilon,\mu) &:= \varepsilon(b_{0010r}c_{0010,1}+b_{0010u}c_{0010,2})\\
c_{0001}(\varepsilon,\mu) &:= c_{0001}(\mu)\\
c_{0100}(\varepsilon,\mu) &:= c_{0100}(\mu)\\
c_{0011}(\varepsilon,\mu) &:= c_{0011}(\mu)+\mathrm{O}(\varepsilon^2)\\
c_{0110}(\varepsilon,\mu) &:= c_{0110}(\mu)+\mathrm{O}(\varepsilon^2)\\
c_{0120}(\varepsilon,\mu) &:= c_{0120}(\mu)+\mathrm{O}(\varepsilon^2)\\
t_{0001}(\varepsilon,\mu) &= 1\\
t_{0100}(\varepsilon,\mu) &= 1\\
t_{0010}(\varepsilon,\mu) &= 1+\varepsilon\lambda_{0010}\\
t_{1000}(\varepsilon,\mu) &= t_{1000}
\end{aligned}$$

where $\mathrm{O}(\varepsilon^2)$ is the linear combination of the gauge cochains having coefficients divisible by ε^2 coming from (higher) cup products, and λ_{0010} is a free parameter.

We plug the universal ε-deformation into the explicit equations in Lemma 6.6. Clearly all the equations are satisfied mod ε. Note that there are 4 free parameters $\lambda,\mu,b_{0010r},b_{0010u}$ (mod ε^2), and three equations $\{r_{0011},r_{0110},r_{0120}\}$. We add one more equation that $\lambda_{0010}=1$ so that the number of parameters matches the number of equations. The mod ε^2 terms of the equations ($=\frac{r_*}{\varepsilon}$ mod ε^2) are given by

$$\left\{\begin{array}{r}
(-2x_{0001})b_{0010r}\mu+(-2s_{0001})b_{0010u}\mu+(2x_{0001})b_{0010r}+(2s_{0001})b_{0010u}+(-s_{0011})\\
(2x_{0100})b_{0010r}\mu^2+(2s_{0100})b_{0010u}\mu^2+(-4x_{0100})b_{0010r}\mu+(-4s_{0100})b_{0010u}\mu+(-2x_{0100})b_{0010r}+(-2s_{0100})b_{0010u}+s_{0110}\mu+(-s_{0110})\\
(-2x_{0110})b_{0010r}\mu+(-2s_{0110})b_{0010u}\mu+(2x_{0110})b_{0010r}+(2s_{0110})b_{0010u}+(-2s_{0120})\\
\lambda_{0010}-1
\end{array}\right\}$$

whose Gröbner basis can be calculated explicitly:

$$\left\{\begin{array}{l}
\mu^2-2\mu+\dfrac{-2x_{0001}s_{0100}s_{0120}-x_{0001}s_{0110}^2+\cdots+x_{0110}s_{0001}s_{0110}+x_{0110}s_{0011}s_{0100}}{2x_{0001}s_{0100}s_{0120}-x_{0001}s_{0110}^2-\cdots+x_{0110}s_{0001}s_{0110}-x_{0110}s_{0011}s_{0100}},\\
b_{0010r}+\dfrac{-4x_{0001}s_{0001}s_{0100}s_{0120}^2+2x_{0001}s_{0001}s_{0110}^2s_{0120}+\cdots+x_{0110}s_{0001}s_{0011}s_{0110}^2-x_{0110}s_{0011}^2s_{0100}s_{0110}}{8x_{0001}^2s_{0100}s_{0110}s_{0120}-8x_{0001}x_{0100}s_{0001}s_{0110}s_{0120}+\cdots-4x_{0100}x_{0110}s_{0001}s_{0011}s_{0110}+4x_{0110}^2s_{0001}s_{0011}s_{0100}}\mu\\
\quad+\dfrac{4x_{0001}s_{0001}s_{0100}s_{0120}^2-2x_{0001}s_{0001}s_{0110}^2s_{0120}-\cdots-x_{0110}s_{0001}s_{0011}s_{0110}^2+x_{0110}s_{0011}^2s_{0100}s_{0110}}{8x_{0001}^2s_{0100}s_{0110}s_{0120}-8x_{0001}x_{0100}s_{0001}s_{0110}s_{0120}+\cdots-4x_{0100}x_{0110}s_{0001}s_{0011}s_{0110}+4x_{0110}^2s_{0001}s_{0011}s_{0100}},\\
b_{0010u}+\dfrac{4x_{0001}^2s_{0100}s_{0120}^2-2x_{0001}^2s_{0110}^2s_{0120}-\cdots-x_{0110}^2s_{0001}s_{0011}s_{0110}+x_{0110}^2s_{0011}^2s_{0100}}{8x_{0001}^2s_{0100}s_{0110}s_{0120}-8x_{0001}x_{0100}s_{0001}s_{0110}s_{0120}+\cdots-4x_{0100}x_{0110}s_{0001}s_{0011}s_{0110}+4x_{0110}^2s_{0001}s_{0011}s_{0100}}\mu\\
\quad+\dfrac{-4x_{0001}^2s_{0100}s_{0120}^2+2x_{0001}^2s_{0110}^2s_{0120}+\cdots+x_{0110}^2s_{0001}s_{0011}s_{0110}-x_{0110}^2s_{0011}^2s_{0100}}{8x_{0001}^2s_{0100}s_{0110}s_{0120}-8x_{0001}x_{0100}s_{0001}s_{0110}s_{0120}+\cdots-4x_{0100}x_{0110}s_{0001}s_{0011}s_{0110}+4x_{0110}^2s_{0001}s_{0011}s_{0100}},\\
\lambda_{0010}-1
\end{array}\right\}$$

The results above are overwhelming both to compute and to decipher. We simplify the procedure using the following lemma:

Lemma 15.5. Let $\operatorname{Spec}A$ be an irreducible affine variety over $\bar{\mathbb{F}}_p$ and let $B=A[s_1,\dots,s_m]/(f_1,\dots,f_m)$. Let $g:\operatorname{Spec}(B)\to\operatorname{Spec}(A)$ be the induced morphism of affine schemes. Suppose there exists a point $\mathfrak{p}\in\operatorname{Spec}(B)$ such that the Jacobian matrix $J=\left(\frac{\partial f_i}{\partial s_j}\right)_{1\leqslant i,j\leqslant m}$ is invertible at $\mathfrak{p}$. Then the image $g(\operatorname{Spec}(B))$ contains an open neighborhood of $g(\mathfrak{p})$. In particular, after replacing both $\operatorname{Spec}A$ and $\operatorname{Spec}B$ by non-empty open subschemes, $\operatorname{Spec}B\to\operatorname{Spec}A$ is étale and surjective.

Proof. Let $\mathcal{U}_B \subseteq \operatorname{Spec}(B)$ be the principal open subscheme where the determinant of the Jacobian matrix J is invertible. By hypothesis, the point $\mathfrak{p}$ lies in $\mathcal{U}_B$. The morphism g restricted to $\mathcal{U}_B$ is étale and thus has open image. □

By Lemma 15.5, we are allowed to assign random values to x_* and s_*, and check the invertibility of Jacobians and the existence of solutions after specializing to that particular set of values of $\{x_*, s_*\}$. For example, if we set

$$\begin{array}{ccccc} s_{0001} = 2 & x_{0001} = 2 & s_{0011} = 4 & x_{0011} = 6 & s_{0100} = 4 \\ x_{0100} = 5 & s_{0110} = 7 & x_{0110} = 3 & s_{0120} = 7 & x_{0120} = 1\text{'} \end{array}$$

then the Gröbner basis becomes

$$G = \{\mu^2 - 2\mu - 15, \qquad b_{0010r}, \qquad b_{0010u} + 1/16\mu - 1/16, \qquad \lambda_{0010} - 1\}$$

and the Jacobian matrix is

$$J = \begin{pmatrix} 2\mu - 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{16} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which has determinant

$$\det(J) = 2(\mu - 1)$$

which is clearly non-trivial modulo G.

By the Newton's method or the multivariable Henselian's lemma, if a polynomial equation has a solution mod ε and the Jacobian is a unit mod ε, then the equation has a solution in $\bar{\mathbb{F}}_p[\![\varepsilon]\!]$.

Require: A configuration (K, Ψ) such that $\Psi_2 \subset \Delta_K$, and a root $\delta_{\mathrm{rot}} \in \Psi_0$ for rotation.

Write down the universal family $\bar{\rho}^{\mathrm{univ}}$ for $X_{\bar{K}_1 \rtimes \breve{T}}$.
Set $\bar{\rho}(\mu) \leftarrow \exp(\mu e_{-\delta_{\mathrm{rot}}})\bar{\rho}^{\mathrm{univ}} \exp(\mu e_{-\delta_{\mathrm{rot}}})$.
Write $t_\alpha(\mu), s_\alpha(\mu), x_{\alpha,i}(\mu)$ for the parameters defining $\bar{\rho}(\mu)$.
Set

$$t_\delta(\varepsilon, \mu) := \begin{cases} t_\delta(\mu) & \delta \notin \Psi_0 \\ 1 + \varepsilon\lambda_\delta & \delta \in \Psi_0 \end{cases}$$

$$x_{\alpha,i}(\varepsilon, \mu) = \begin{cases} x_{\alpha,i}(\mu) & \alpha \notin \Psi_0 \\ \varepsilon b_{\alpha,i} & \alpha \in \Psi_0 \end{cases}$$

$$s_\alpha(\varepsilon, \mu) = s_\alpha(\mu),$$

and treat

$$\{\lambda_*, \mu, b_{*,*}\}$$

as the free variables.
Plug $t_\alpha(\varepsilon, \mu), s_\alpha(\varepsilon, \mu), x_{\alpha,i}(\varepsilon, \mu)$ into the equations $\{\frac{r_\alpha}{\varepsilon} : \alpha \in \Psi_2, \mathrm{ht}(\alpha) > 1\}$ and write P_α for the constant terms (with respect to ε).
$G \leftarrow$ Gröbner basis of $\{P_\alpha : \alpha \in \Psi_2, \mathrm{ht}(\alpha) > 1\} \cup \{\sum_\delta \lambda_\delta - 1\}$ over the function field of $X_{\bar{K}_1 \rtimes \breve{T}}$.
if $G = \{1\}$ **then** ▷ There is no mod ε solution.
    Report failure!
end if
$G_{\mathrm{aug}} \leftarrow$ Gröbner basis of $\{P_\alpha : \alpha \in \Psi_2, \mathrm{ht}(\alpha) > 1\} \cup \{\sum_\delta \lambda_\delta - 1, \text{Jacobian of } \{P_\alpha\}\}$.
if $G = G_{\mathrm{aug}}$ **then** ▷ Jacobian is not a unit.
    Report failure!
end if
Report success.

Algorithm 12. The Rotational Deformational Test

Theorem 15.6. Each of the Configurations I-IV pass the Rotational Deformation Test (c.f. Algorithm 12) by using the following rotation roots:

(I, II, IV) $\delta_{\mathrm{rot}} = 0010$,
(III) $\delta_{\mathrm{rot}} = 1000$.

Proof. This follows from the exact computation in Appendix C. We remark that for Configuration IV, there are three choices $\delta_{\mathrm{rot}} \in \{1000, 0010, 0001\}$, but only 0010 works; similarly for Configuration III, there are two choices $\delta_{\mathrm{rot}} \in \{1000, 0010\}$, but only 1000 works. ☐

Remark 15.7. We don't exclude the possibility that for more complex groups like E_6, we need to rotate using multiple roots $\{\delta_{\mathrm{rot}}\} \subset \Psi_0$.

Corollary 15.8. When $\breve{G} = \mathrm{F}_4$, $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ is equidimensional of dimension $[F : \mathbb{Q}_p]\#\Phi^+$ and has irreducible components identified with the irreducible components of $\mathcal{W}_{\breve{B}}$.

16. The existence of crystalline lifts

We use the integral fixed-type stacks of [Lin25]. The crystalline stack is obtained by imposing the closed condition $N = 0$ on the stack of Breuil–Kisin–Fargues Gal_F-modules. Thus the proofs of [Lin25, Proposition B.5.10 and Theorem B.6.3] apply without change.

Lemma 16.1. *Fix a bounded Hodge type and inertial type. The morphism from the stack of crystalline Breuil–Kisin–Fargues lattices of that type to the integral Emerton–Gee stack is proper. Its scheme-theoretic image is $\mathbb{Z}_p$-flat, and a geometric point of its special fiber belongs to the image if and only if the corresponding residual representation has a crystalline lift of that type.*

Proof. This is the $N = 0$ case of [Lin25, Proposition B.5.10 and Theorem B.6.3]. The only difference is that we use the closed crystalline substack. □

Lemma 16.2. Suppose $\breve{G} = \mathrm{PGL}_2$. There exist finitely many Hodge types $\{\underline{\lambda}_s | s \in S\}$ such that the special fiber of the crystalline stack $\bigcup_{s\in S} \mathcal{X}^{\underline{\lambda}_s,\tau_{\mathrm{triv}}}_{\mathrm{PGL}_2}$ contains all irreducible components of $\mathcal{X}^{\mathrm{EG}}_{\mathrm{PGL}_2,\mathrm{red}}$. Here, τ_{triv} stands for the trivial inertial type. The lemma also holds after replacing $\breve{G}$ by its Borel.

Proof. This is [EG23, Theorem 6.4.4]. □

Lemma 16.3 (Borel crystalline assembly)**.** *Let $C \subset \mathcal{W}_{\breve{B}}$ be irreducible. There is a regular Borel Hodge type $\underline{\lambda}_C$ with trivial inertial type such that the crystalline special fiber contains the component $\overline{\dot{\mathcal{X}}_{\breve{B}}(C)}$.*

Proof. Apply Lemma 16.2 to the simple roots and choose a sufficiently regular common Hodge type. Then use [Lin25, Corollary 5 and the proof of Lemma 19]. Finally apply Lemma 16.1. □

Corollary 16.4. Suppose $\breve{G}$ is arbitrary.

(1) There exist finitely many Hodge types $\{\underline{\lambda}_s | s \in S\}$, such that the special fiber of the crystalline stack $\bigcup_{s\in S} \mathcal{X}^{\underline{\lambda}_s,\tau_{\mathrm{triv}}}_{\breve{B}}$ maps surjectively on the Weil-Deligne stack $\mathcal{W}_{\breve{B}}$.

(2) The special fiber of the crystalline stack $\bigcup_{s\in S} \mathcal{X}^{\underline{\lambda}_s,\tau_{\mathrm{triv}}}_{\breve{B}}$ contains $\overline{\dot{\mathcal{X}}_{\breve{B}}(C)}$ for all irreducible components $C \subset \mathcal{W}_{\breve{B}}$.

Proof. (1) Apply Lemma 16.3 to every irreducible component of the finite-type stack $\mathcal{W}_{\breve{B}}$ and take the finite set of resulting Hodge types. Each component is dominated by the indicated crystalline special fiber, giving the asserted surjection.

(2) The reduced special fiber of $\bigcup_{s\in S} \mathcal{X}^{\underline{\lambda}_s,\tau_{\mathrm{triv}}}_{\breve{B}}$ is a union $\bigcup_{\mathcal{C}\in S_{\mathcal{C}}} \mathcal{C}$ of (top-dimensional) irreducible components of the special fiber of the corresponding $\mathcal{X}_{\breve{B}}$-stack that maps surjectively onto $\mathcal{W}_{\breve{B}}$.

For each irreducible component $C \subset \mathcal{W}_{\breve{B}}$, there exists a unique irreducible component $C_{\mathcal{X}}$ of $\mathcal{X}_{\breve{B}}$ (necessarily of the form $\overline{\dot{\mathcal{X}}_{\breve{B}}(C)}$) satisfying the condition that $C_{\mathcal{X}} \to C$ is scheme-theoretically dominant. Thus $C_{\mathcal{X}} \in S_{\mathcal{C}}$ for all C by part (1). □

Theorem 16.5. *If $\breve{G} = \mathrm{F}_4$, then all $\mathrm{Gal}_F \to \breve{G}(\bar{\mathbb{F}}_p)$ have a crystalline lift.*

If $\breve{G}$ is a reductive group satisfying the standing hypotheses and $[F : \mathbb{Q}_p] > \frac{\dim \breve{G}}{2}$, then all $\mathrm{Gal}_F \to \breve{G}(\bar{\mathbb{F}}_p)$ have a crystalline lift.

Proof. First suppose that the semisimple rank is at least two. Since $\dim \breve{G} = 2\#\Phi^+ + \operatorname{rank} \breve{G}$, the strict inequality $[F : \mathbb{Q}_p] > \dim \breve{G}/2$ implies $[F : \mathbb{Q}_p] \geqslant \#\Phi^+ + 2$. Proposition 14.2 then shows that every irreducible component of $\mathcal{X}^{\mathrm{EG}}_{\breve{G},\mathrm{red}}$ is one of the components dominating a component of $\mathcal{W}_{\breve{B}}$. Corollary 16.4 places each of these components in the special fiber of a crystalline stack. Every point lies on an irreducible component and hence in that special fiber; by the pointwise characterization in Lemma 16.1, it has a crystalline lift.

For a rank-one factor the same assertion is Corollary 16.4, based on Lemma 16.2; for a torus it is the corresponding rank-one character statement. For a general reductive group, choose the Hodge type factor by factor using fundamental coweights and take their Cartesian product. This construction is compatible with the central torus and therefore gives a single Hodge type for the original group.

When $\breve{G} = \mathrm{F}_4$, use Corollary 15.8 in place of the large-degree application of Proposition 14.2, and conclude in the same way from Corollary 16.4. $\square$

Appendix A. Bounded cone and F_4 strata algorithms

This appendix gives the exhaustive certification. Root data are generated from the F_4 Cartan matrix. The reviewed fine-pair list is checked directly and reconstructed when absent. Roots are coefficient tuples in Bourbaki order, and all rank calculations are exact.

A.1. **Field-degree bound.** For each of the 105 downward-closed complements K, the program builds the complete grouped cone matrix for the largest possible target set. An integral nonzero minor certifies its rank. The resulting dimension estimate is affine in $d = [F : \mathbb{Q}_p]$ and is automatic for $d \geqslant 2$. The contents of all witness minors have prime support contained in $\{2, 3\}$, so the estimate is unchanged in the standing range $p > 12$.

Require: The F_4 Cartan matrix.
Generate Φ^+, its root order, and all 105 complements K.
for each K **do**
  Build the full grouped cone matrix at $\Psi_2 = \Phi^+$.
  Find an integral maximal nonzero minor and its bad primes.
  Solve the resulting affine dimension inequality for d.
end for
Return the largest exceptional degree range.

Algorithm 13. A priori field-degree bound

Thus only $d = 1$ requires enumeration.

A.2. **Fine pairs.** At degree one, a candidate level set is a subset $S \subset \Phi^+$ on which a linear functional can satisfy $f(\alpha) = 1$ for every $\alpha \in S$. Every such S contains a linearly independent subset with the same affine span. This gives exhaustive small seeds and avoids scanning all 2^{24} subsets. We keep the resulting list of 4862 fine pairs.

Require: Φ^+, the degree bound $d = 1$, and the reviewed pair list P.
if P is absent **then**
  Generate the independent affine-consistent seeds.
  Expand them by BFS, stopping at affine inconsistency or the degree bound, and write the resulting canonical list P.
end if
for $(\Psi_0, \Psi_2) \in P$ **do**
  Recompute the difference lattice, Ψ_0, and the closure axioms.
  Regard the pair as a terminal BFS state; do not generate children.
end for
for each independent subset of the required degree **do**
  Check that its fine pair occurs in P.
end for
Return P.

Algorithm 14. Fine-pair certification and fallback BFS

There are 4862 distinct fine pairs. All 3002 relevant independent subsets occur in this list. The cutoff is sound because adjoining a root to Ψ_2 strictly increases minimal degree.

A.3. **Exhaustive classification.** We apply the cone calculation to the 105 possibilities for K and the 4862 fine pairs. Exactly four cases have $\varepsilon = 0$; these are Configurations I–IV. All integral rank witnesses remain nonzero for $p > 12$.

For the four displayed grouped cone matrices the tuples (#variables, #relations, rank, dimension) are

$$(28, 17, 17, 11), \quad (29, 17, 17, 12), \quad (30, 14, 14, 16), \quad (32, 8, 7, 25).$$

Relations with the same target are one summed equation, never separate ideal generators. The implementation details and reviewed data are available at https://github.com/mocham/AlgEG/tree/main/code/v4.

Appendix B. Root-poset cycle algorithms

We describe separately the absolute and non-Borel computations. In both, a source-leaf contraction removes a source of valency one together with its unique target; isolated sources are discarded. A surviving Jacobian with r target rows and $c \geqslant r$ source columns is tested by enumerating the $\binom{c}{r}$ source-column subsets. Success means that one corresponding determinant is a nonzero monomial in the simple-root variables.

B.1. **Absolute root posets.**

The absolute routine applies the following finite test independently at each height.

Require: An exceptional root system Φ and its Chevalley constants.
Construct the directed root poset and its bipartite height layers $\mathfrak{B}(\Phi^+, \Delta, h)$.
for each height h and each target selection T **do**
    Retain the full source layer and only edges ending in T.
    Repeatedly contract source leaves.
    if the graph contracts to the empty graph **then**
        Record a trivial certificate and continue.
    end if
    Form the Jacobian $J_{\alpha,\beta} = N_{\beta,\alpha-\beta} x_{\alpha-\beta}$.
    for each set C of $\#T$ source columns **do**
        Compute $d_C \leftarrow \det J_C$.
        if d_C is a nonzero monomial **then**
            Record C and d_C and continue with the next target selection.
        end if
    end for
    If no witness was found, report failure.
end for

Algorithm 15. Absolute admissibility test

For the exceptional types the numerical summary is:

Type	target selections	failures
F_4	48	0
E_6	138	0
E_7	384	0
E_8	1196	0

The nontrivial determinants, up to orientation-dependent signs, are:

Type	height	Jacobian minor
F_4	3	$3x_{0001}x_{0010}x_{1000}$
E_6	2	$2x_{000010}x_{001000}x_{010000}$
E_6	3	$3x_{000001}x_{000010}x_{001000}x_{010000}x_{100000}$
E_7	2	$-2x_{0000100}x_{0010000}x_{0100000}$
E_7	3	$-3x_{0000010}x_{0000100}x_{0010000}x_{0100000}x_{1000000}$
E_7	4	$2x_{0000001}x_{0001000}x_{0010000}x_{0100000}$
E_7	8	$-2x_{0000001}x_{0000010}x_{0000100}x_{0010000}$
E_8	2	$-2x_{00001000}x_{00100000}x_{01000000}$
E_8	3	$-3x_{00000100}x_{00001000}x_{00100000}x_{01000000}x_{10000000}$
E_8	4	$2x_{00000010}x_{00010000}x_{00100000}x_{01000000}$
E_8	5	$-5x_{00000001}x_{00000010}x_{00000100}x_{00010000}x_{00100000}x_{01000000}x_{10000000}$
E_8	8	$-2x_{00000010}x_{00000100}x_{00001000}x_{00100000}$
E_8	9	$3x_{00000001}x_{00000100}x_{00001000}x_{00010000}x_{00100000}x_{10000000}$
E_8	14	$2x_{00000010}x_{00001000}x_{00010000}x_{01000000}$

For D_n, all cycles are isomorphic to $\vec{C}_6$. With the standard ordering $\Delta_{D_n} = \{\delta_1, \ldots, \delta_n\}$, they are given by

$$\beta_1 = \sum_{i=n-h-1}^{n-2} \delta_i, \quad \beta_2 = \sum_{i=n-h}^{n-2} \delta_i + \delta_{n-1}, \quad \beta_3 = \sum_{i=n-h}^{n-2} \delta_i + \delta_n,$$

and their Jacobian coefficients are 2 or -2.

B.2. **Non-Borel root posets.** Theorem 10.11 already has the correct disjunction: a relative cycle is either non-obstructing or its associated absolute graph is admissible. Thus the non-obstructing condition is applied before any absolute Jacobian test.

Require: A Levi $\breve{M}$, an elliptic Weyl element w, its relative root orbits, and the non-obstructing singleton and pair relation.

Construct the w-relative root poset and its bipartite height layers.
Remove individually non-obstructing targets.
for each relative height layer **do**
  Contract source leaves.
  for each target subset T containing no non-obstructing pair **do**
    Restrict to T and contract source leaves again.
    if the graph contracts to the empty graph **then**
      Record the contraction and continue.
    end if
    Replace every relative source and target by all absolute roots in its w-orbit.
    Add every absolute edge whose target-source difference is a positive root, and contract source leaves.
    if the absolute graph is nonempty **then**
      Apply Algorithm 15, testing combinations of absolute source columns.
    end if
  end for
end for
Report failure if any surviving absolute graph has no monomial maximal minor.

Algorithm 16. Non-Borel cycle test

The computation generates all exceptional Levi/Weyl configurations from the Cartan matrices and covers:

Type	configurations	failures
F_4	19	0
E_6	68	0
E_7	145	0
E_8	304	0

Consequently all F_4 relative cycles are non-obstructing or contractible, and every surviving E-type associated absolute graph is admissible. The reviewed data are available at https://github.com/mocham/AlgEG/tree/main/code/v4.

Appendix C. Characteristic-uniform rotation certificates

The computation is over $\mathbb{Q}$ and records enough integral data to spread each smooth point to every characteristic $p > 12$.

For a rotation root δ_{rot} and $\alpha \in \Psi_2$, the code follows the complete string $\alpha + n\delta_{\mathrm{rot}}$. At step n it multiplies the current coefficient by $N_{-\delta_{\mathrm{rot}},\alpha+n\delta_{\mathrm{rot}}}/n$. Thus, for Configuration I,

$$c_{0100} \mapsto c_{0100} + 2\mu c_{0110} - \mu^2 c_{0120}, \qquad c_{0001} \mapsto c_{0001} - \mu c_{0011}.$$

Require: (K, Ψ_0, Ψ_2) and a rotation root δ_{rot}.

Generate all Chevalley constants and root-string substitutions.
Form the divided ε-deformation equations over $\mathbb{Q}$.
for the deterministic sequence of small rational specializations and points **do**
    Evaluate every equation exactly.
    Evaluate the square Jacobian determinant exactly.
    if the equations vanish and the determinant is nonzero **then**
        Factor all denominators, required nonzero coordinates, and the determinant; record their prime support and stop.
    end if
end for
Fail unless the recorded bad primes are all at most 12.

Algorithm 17. Rational rotation witness

The exact output is:

Configuration	δ_{rot}	$\det J$	bad primes
I	0010	432	$2, 3$
II	0010	704	$2, 3, 5, 11$
III	1000	15360	$2, 3, 5$
IV	0010	895795200	$2, 3, 5$

Every equation evaluates to zero at its recorded rational point. Inverting the displayed finite set makes that point a smooth section, so reduction gives a smooth point in every characteristic $p > 12$. The reviewed data are available at https://github.com/mocham/AlgEG/tree/main/code/v4.

Appendix D. Computational source

The implementation and reviewed data are available at

https://github.com/mocham/AlgEG/tree/main/code/v4.

The code generates the root systems, Chevalley constants, order ideals, elliptic Weyl orbits, and twisted configurations used in the preceding appendices. The fine-pair list is checked as in Algorithm 14; it is reconstructed from first principles only when the reviewed data are absent.

The precise source map, commands, runtimes, certificate formats, checksums, and artifact inventory are recorded in

https://github.com/mocham/AlgEG/blob/main/code/README.md.

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