HEISENBERG VARIETIES AND THE EXISTENCE OF DE RHAM LIFTS

ZHONGYIPAN LIN

Abstract. Let F be a p-adic field. For certain non-abelian nilpotent algebraic groups U over $\bar{\mathbb{Z}}_p$ equipped with Gal_F-action, we study the associated *Heisenberg varieties* which model the non-abelian cohomology set "$H^1(\mathrm{Gal}_F, U)$". The construction of Heisenberg varieties involves the Herr complexes including their cup product structure.

Write U_n for a quasi-split unitary group and assume $p \neq 2$. We classify mod p Langlands parameters for U_n (quasi-split), SO_{2n+1}, SO_{2n}, Sp_{2n}, GSpin_{2m} and GSpin_{2m+1} (split) over F, and show they are successive Heisenberg-type extensions of elliptic Langlands parameters.

We employ the Heisenberg variety to study the obstructions for lifting a non-abelian cocycle along the map $H^1(\mathrm{Gal}_F, U(\bar{\mathbb{Z}}_p)) \to H^1(\mathrm{Gal}_F, U(\bar{\mathbb{F}}_p))$. We present a precise theorem that reduces the task of finding de Rham lifts of mod p Langlands parameters for unitary, symplectic, orthogonal, and spin similitude groups to the dimension analysis of specific closed substacks of the reduced Emerton-Gee stacks for the corresponding group.

Finally, we carry out the dimension analysis for the unitary Emerton-Gee stacks using the geometry of Grassmannian varieties. The paper culminates in the proof of the existence of potentially crystalline lifts of regular Hodge type for all mod p Langlands parameters for p-adic (possibly ramified) unitary groups U_n. It is the first general existence of de Rham lifts result for non-split (ramified) groups, and provides evidence for the topological Breuil-Mézard conjecture for more general groups.

Contents

1. Introduction

Let G be a reductive group over a p-adic field F which splits over a tame extension K/F, and let ${}^L G = \widehat{G} \rtimes \mathrm{Gal}(K/F)$ be the Langlands dual group of G.

In our previous work [L23], we classified elliptic mod p Langlands parameters for G and constructed their de Rham lifts.

In this paper, we shift our attention to parabolic mod p Langlands parameters. Let $P \subset \widehat{G}$ be a $\mathrm{Gal}(K/F)$-stable parabolic. Write ${}^LP$ for $P \rtimes \mathrm{Gal}(K/F)$. A mod p Langlands parameter is either elliptic, or factors through some ${}^LP$.

Let $\bar{\rho} : \mathrm{Gal}_F \to {}^LP(\bar{\mathbb{F}}_p)$ be a parabolic mod p Langlands parameter. We are interested in the following question:

Question: Does there exist a de Rham lift $\rho : \mathrm{Gal}_F \to {}^LP(\bar{\mathbb{Z}}_p)$ of regular Hodge type?

This question is addressed for $G = \mathrm{GL}_n$ in the book by Emerton and Gee, and has important applications to the geometric Breuil-Mézard conjecture(see [EG23] and [LLHLM23]).

We briefly describe the general strategy employed in [EG23] and explain how the proof breaks for groups which are not GL_n. Let ${}^LM$ denote a $\mathrm{Gal}(K/F)$-stable Levi subgroup of ${}^LP$ and write U for the unipotent radical of P. Write $\bar{\rho}_M : \mathrm{Gal}_F \xrightarrow{\bar{\rho}} {}^LP(\bar{\mathbb{F}}_p) \to {}^LM(\bar{\mathbb{F}}_p)$ for the ${}^LM$-semisimplification of $\bar{\rho}$. The construction ρ follows a 2-step process:

- Step 1: Carefully choose a lift $\rho_M : \mathrm{Gal}_F \to {}^LM(\bar{\mathbb{Z}}_p)$ of $\bar{\rho}_M$.
- Step 2: Endow $U(\bar{\mathbb{Z}}_p)$ with the Gal_F-action induced by ρ_M. Show that the image of

$$H^1(\mathrm{Gal}_F, U(\bar{\mathbb{Z}}_p)) \to H^1(\mathrm{Gal}_F, U(\bar{\mathbb{F}}_p))$$

  contains the cocycle corresponding to $\bar{\rho}$.

1.1. Partial lifts after abelianization and the work of Emerton-Gee

Let $[U, U]$ denote the derived subgroup of U and write $U^{\mathrm{ab}} := U/[U, U]$ for the abelianization of U.

The first approximation of a de Rham lift of $\bar{\rho}$ is a continuous group homomorphism

$$\mathrm{Gal}_F \to \frac{{}^LP}{[U, U]}(\bar{\mathbb{Z}}_p)$$

lifting $\bar{\rho}$ modulo $[U, U]$. The groundbreaking idea presented in [EG23] is that we can achieve the following:

- Step 1: Choose a lift $\rho_M : \mathrm{Gal}_F \to {}^LM(\bar{\mathbb{Z}}_p)$ of $\bar{\rho}_M$.
- Step 2: Guarantee the image of

$$H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{Z}}_p)) \to H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p))$$

  contains the cocycle corresponding to $\bar{\rho} \mod [U, U]$,

as long as we can estimate the dimension of certain substacks of the reduced Emerton-Gee stacks for M.

We formalize the output of their geometric argument as follows:

Property EPL. *(Existence of partial lifts) Let* $\operatorname{Spec} R$ *be a non-empty potentially crystalline deformation ring of* $\bar{\rho}_M$ *such that for some* $x \in \operatorname{Spec} R(\bar{\mathbb{Q}}_p)$, $H^1_{\mathrm{crys}}(\mathrm{Gal}_K, U^{\mathrm{ab}}(\bar{\mathbb{Q}}_p)) = H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\bar{\mathbb{Q}}_p))$. *Then there exists a point* $y \in \operatorname{Spec} R(\bar{\mathbb{Z}}_p)$ *such that the image of* $H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{Z}}_p)) \to H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p))$ *contains the cocycle corresponding to* $\bar{\rho} \mod [U, U]$.

The geometric input is as follows:

Property SSD. *(Sufficiently small dimension) Write $\mathcal{X}_{F,{}^LM,\mathrm{red}}$ for the reduced Emerton-Gee stacks for M and write $X_s \subset \mathcal{X}_{F,{}^LM,\mathrm{red}}$ for the (scheme-theoretic closure of the) locus*

$$\{x| \dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p)) \geq s\}.$$

Then

$$\dim X_s + s \leq [F:\mathbb{Q}_p] \dim M/B_M$$

where B_M is a Borel of M.

Here we define the dimension of an empty set to be $-\infty$ to avoid confusion.

Theorem 1. *Property SSD implies Property EPL.*

Proof. The proof of [EG23, Theorem 6.1.1, Theorem 6.3.2] works verbatim. See also [L21, Theorem 5.1.2]. The proof needs the algebraicity of the reduced Emerton-Gee stacks, which is established in [L23B], as well as the basic properties of the potentially crystalline deformation rings, which are the main results of [BG19]. □

For $G = \mathrm{GL}_n$, we have $F = K$ and we can choose P such that $U = U^{\mathrm{ab}}$. However, for general groups G, we immediately run into the problem that $U \neq U^{\mathrm{ab}}$.

1.2. Heisenberg-type extensions

The good news is that for classical groups, we can always choose P such that U is the next best thing after abelian groups, namely, unipotent algebraic groups of nilpotency class 2.

Theorem 2. *(Lemma 7.2, Lemma 8.1, Lemma 9.1) Let ${}^LG$ be any of ${}^LU_n, \mathrm{GSp}_{2n}$ or GSO_n.*

Then each mod p Langlands parameter $\bar{\rho} : \mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$ is either elliptic, or factors through a maximal proper parabolic ${}^LP$ such that $\bar{\rho}$ is a Heisenberg-type extension (see Definition 5.1) of some $\bar{\rho}_M : \mathrm{Gal}_F \to {}^LM(\bar{\mathbb{F}}_p)$ where ${}^LM$ is the Levi factor of ${}^LP$.

A *Heisenberg-type extension* is, roughly speaking, an extension which has the least amount of "non-linearity". More precisely, if $\bar{\rho}_P : \mathrm{Gal}_F \to {}^LP(\bar{\mathbb{F}}_p)$ is a Heisenberg-type extension of $\bar{\rho}_M$, then $[U,U]$ is an abelian group, and

$$\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p)) \leq 1.$$

The key technical result of this paper is Theorem 5.5. Roughly speaking, for Heisenberg-type extensions, the non-linear part of the obstruction for lifting is so mild that it can be killed through manipulating cup products. To make this idea work, we need a resolution of Galois cohomology supported on degrees $[0,2]$, which is compatible with cup products on the cochain level. In this paper, the resolution used is the Herr complexes. Although Herr complexes are infinite-dimensional resolutions, we can truncate them to a finite system while still retaining the structure of cup products. The Heisenberg equations are defined through cup products on the truncated Herr cochain groups. The main technical work is done in Section 1-5. In Section 6-9, we study unitary groups, symplectic groups and orthogonal groups on a case-by-case basis and prove the following.

Theorem 3. *Let ${}^LG_n$ be any of ${}^LU_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{GSp}_{2n}$ or GSO_n, and assume $p \neq 2$.*

Assume for each n and each maximal proper Levi ${}^LM$ of ${}^LU_n$, Property SSD holds for ${}^LM$.

Then all L-parameters $\mathrm{Gal}_F \to {}^LG_n(\bar{\mathbb{F}}_p)$ admit a potentially crystalline lift of regular Hodge type.

We remark that GSp_{2n} and GSO_{2n} are the Langlands dual groups of the spin similitude groups, while Sp_{2n}, SO_{2n} and SO_{2n+1} are the Langlands dual groups of SO_{2n+1}, SO_{2n} and Sp_{2n}, resp..

1.3. The Emerton-Gee stacks for unitary groups

We prove the following theorem.

Theorem 4. *(Theorem 10.1) Let $\bar{\alpha}:\mathrm{Gal}_K\to\mathrm{GL}_a(\bar{\mathbb{F}}_p)$ be an irreducible Galois representation.*

The locus of $\bar{x}\in\mathcal{X}_{F,{}^LU_n,\mathrm{red}}$ such that $\mathrm{Hom}_{\mathrm{Gal}_K}(\bar{\alpha},\bar{x}|_{\mathrm{Gal}_K})\geq r$ is of dimension at most $[F:\mathbb{Q}_p]\frac{n(n-1)}{2}-r^2+\frac{r}{2}$.

Since Theorem 4 is stronger than Property SSD, we have established the existence of de Rham lifts for U_n.

Theorem 5. *If $p\neq 2$, all L-parameters $\mathrm{Gal}_F\to{}^LU_n(\bar{\mathbb{F}}_p)$ admits a potentially crystalline lift of regular Hodge type.*

Proof. Note that $H^2(\mathrm{Gal}_F,U^{\mathrm{ab}}(\bar{\mathbb{F}}_p))\cong H^2(\mathrm{Gal}_K,\bar{\alpha}\otimes\bar{x}|^\vee_{\mathrm{Gal}_K})$ and $\lfloor -r^2+r/2\rfloor\leq -r$ for all $r\geq 1$. □

The proof of Theorem 4 is much more involved compared to its GL_n-analogue worked out in [EG23]. The GL_n-case only requires the computation of the rank of certain vector bundles, and is a completely linear problem. For U_n, we need to compute the relative dimension of certain quadratic cones.

To prove the required bound for U_n, we need a very precise control of the rank of cup products for extensions of the form $\begin{bmatrix}\bar{\alpha}(1)^{\oplus r} & * & * \\ & \bar{\tau} & * \\ & & \bar{\alpha}^{\oplus r}\end{bmatrix}$, and in order to do that, we need to relate the rank of cup products to the dimension of Grassmannian manifolds (with the key lemma being 10.10). The inequality is extremely tight at many steps. After we have obtained an estimate for the rank of cup products, we still need to divide the question into multiple cases and perform min-max optimization on multi-variable polynomial functions in each of these cases.

The method of proof we presented in this paper also applies to orthogonal/symplectic/spin similitude groups. However, we do not treat these groups in this paper due to the complexity of the analysis involved.

1.4. Final remarks

In the literature, the geometric Breuil-Mézard conjecture is often proved for sufficiently generic tame inertial types (for example, see [LLHLM23]).

For GL_d, if we only care about the generic situation, then the existence of de Rham lifts is straightforward. Let $\bar{r}=\begin{bmatrix}\bar{r}_1 & * & \dots & * \\ & \bar{r}_2 & \dots & * \\ & & \dots & * \\ & & & \bar{r}_m\end{bmatrix}$ be a Galois representation which is maximally non-split, meaning it factors through a unique minimal parabolic. If $\bar{r}_i(1)\neq\bar{r}_{i+1}$ for each i, then $\bar{r}$ admits a crystalline lift for trivial reasons. Indeed, put $\bar{r}^i:=\begin{bmatrix}\bar{r}_i & * & \dots & * \\ & \bar{r}_{i+1} & \dots & * \\ & & \dots & * \\ & & & \bar{r}_m\end{bmatrix}$; once an arbitrary lift r^i of $\bar{r}^i$ is chosen, we can construct a lift of r^{i-1} as an extension of r^i and a lift of $\bar{r}_i$.

However, for general groups such as the unitary groups, regardless of how generic the situation is, we do not have an easy way of constructing de Rham lifts. The reason being that "maximal non-splitness" is not very useful for general groups. Although it remains a strong constraint, it is not easy to be directly utilized. Consider the symplectic similitude group situation for the sake of reusing notations.

The argument in the previous paragraph breaks unless $\bar{r}_i(1) \neq \bar{r}_j$ for all i, j. Put $\bar{\tau} := \begin{bmatrix} \bar{r}_2 & \dots & * \\ & \dots & * \\ & & \bar{r}_{m-1} \end{bmatrix}$ and thus $\bar{\rho} = \begin{bmatrix} \bar{r}_1 & \bar{c}_1 & \bar{c}_3 \\ & \bar{\tau} & \bar{c}_2 \\ & & \bar{r}_m \end{bmatrix}$. Suppose we have chosen a lift (r_1, τ, r_m) of $(\bar{r}_1, \bar{\tau}, \bar{r}_m)$. A lift c_1 of $\bar{c}_1$ uniquely determines a lift c_2 of $\bar{c}_2$ and vice versa. Let's choose a c_1 and, thus, a c_2. Now we run into the problem that a lift c_3 of $\bar{c}_3$ does not exist for all choices of c_1. For a lift c_3 to exist, we must have $c_1 \cup c_2 = 0$, which is a non-linear condition. Even if $c_1 \cup c_2 = 0$, we can only ensure there exists a c_3 which makes $\begin{bmatrix} r_1 & c_1 & c_3 \\ & \tau & c_2 \\ & & r_m \end{bmatrix}$ a group homomorphism; there is no guarantee that c_3 lifts $\bar{c}_3$! The obstruction disappears if $\bar{r}_1(1) \neq \bar{r}_m$; but such restrictions will force the Serre weights to lie within very narrow strips of a chosen alcove.

From this perspective, Theorem 5 is necessary even if we only aim to prove the Breuil-Mézard conjecture in the generic situation.

2. Heisenberg equations

Let $r, s, t \in \mathbb{Z}_+$ be integers. Let Λ be a DVR with uniformizer ϖ. Let $d \in \mathrm{Mat}_{s\times t}(\Lambda)$ and $\Sigma_1, \dots, \Sigma_s \in \mathrm{Mat}_{r\times r}(\Lambda)$ be constant matrices. For ease of notation, for $x \in \mathrm{Mat}_{r\times 1}(\Lambda)$, write

$$x^t\Sigma x := \begin{bmatrix} x^t\Sigma_1 x \\ \dots \\ x^t\Sigma_s x \end{bmatrix} \in \mathrm{Mat}_{s\times 1}(\Lambda).$$

Here x^t denotes the transpose of x. We are interested in solving systems of equations in $(r+t)$ variables of the form

$$(\dagger) \qquad x^t\Sigma x + dy = 0$$

where $x \in \Lambda^{\oplus r}$ and $y \in \Lambda^{\oplus t}$ are the $(r+t)$ variables. We will call $(\dagger)$ the quadratic equation with coefficient matrix (Σ, d).

2.1. Lemma Let Λ be a DVR with uniformizer ϖ. Let M be a finite flat Λ-module. If $N \subset M$ is a submodule such that $M/N \cong \Lambda/\varpi^n$ $(n > 0)$, then there exists a Λ-basis $\{x_1, x_2, \dots, x_s\}$ of M such that $N = \mathrm{span}(\varpi^n x_1, x_2, \dots, x_s)$.

Proof. Let $\{e_1, \dots, e_s\}$ be a Λ-basis of M and let $\{f_1, \dots, f_s\}$ be a Λ-basis of N. There exists a matrix $X \in \mathrm{GL}_s(\Lambda[1/\varpi])$ such that $(f_1|\dots|f_s) = X(e_1|\dots|e_s)$. By the theory of Smith normal form, $X = SDT$ where $S, T \in \mathrm{GL}_s(\Lambda)$ and D is a diagonal matrix. We have $D = \mathrm{Diag}(\varpi^n, 1, \dots, 1)$. Set $(x_1|\dots|x_s) := T(e_1|\dots|e_s)$, and we are done. □

2.2. Definition A quadratic equation with coefficient matrix (Σ, d) is said to be *Heisenberg* if

(H1) $\mathrm{coker}\, d \cong \Lambda$ or Λ/ϖ^n; and
(H2) there exists $f \in \Lambda^{\oplus r}$ such that $f^t\Sigma f \neq 0 \bmod (\varpi, \mathrm{Im}(d))$.

2.3. Theorem Let (Σ, d) be a Heisenberg equation over Λ. If there exists a mod ϖ solution $(\bar{x}, \bar{y}) \in (\Lambda/\varpi)^{\oplus r+t}$ to (Σ, d) (that is, $\bar{x}^t\Sigma\bar{x} + d\bar{y} \in \varpi\Lambda^{\oplus s}$), then there exists an extension of DVR $\Lambda \subset \Lambda'$ such that there exists a solution $(x, y) \in \Lambda'^{\oplus r+t}$ of (Σ, d) lifting $(\bar{x}, \bar{y})$.

Proof. Write $\{e_1, \dots, e_s\}$ for the standard basis for $\Lambda^{\oplus s}$. By Lemma 2.1, we can assume $\mathrm{Im}(d) = \mathrm{span}(\varpi^n e_1, e_2, \dots, e_s)$ or $\mathrm{span}(e_2, \dots, d_s)$. Write $d = \begin{bmatrix} d_1 \\ \dots \\ d_s \end{bmatrix}$. By Definition 2.2, there exists an element $f \in \Lambda^{\oplus r}$ such that $f^t\Sigma f \neq 0 \mod (\varpi, \mathrm{Im}(d))$; equivalently, $f^t\Sigma_1 f \neq 0 \mod \varpi$. Let (x, y) be an arbitrary lift of $(\bar{x}, \bar{y})$. Let Λ' be the ring of integers of the algebraic closure of $\Lambda[1/\varpi]$. Let $\lambda \in \Lambda'$. Consider

$$(x + \lambda f)^t\Sigma_1(x + \lambda f) + d_1 y \quad = (f^t\Sigma_1 f)\lambda^2 + (x^t\Sigma_1 f + f^t\Sigma_1 x)\lambda + (x^t\Sigma_1 x + d_1 y);$$

note that the ϖ-adic valuation of the quadratic term is 0 while the ϖ-adic valuation of the constant team is positive. By inspecting the Newton polygon, the quadratic equation above admits a solution $\lambda \in \Lambda'$ of positive ϖ-adic valuation. By replacing x by $x + \lambda f$, we can assume

$$x^t\Sigma_1 x + d_1 y = 0.$$

Equivalently, $x^t\Sigma x + dy \in \mathrm{span}(e_2, \dots, e_s) \subset \mathrm{Im}(d)$. By replacing Λ by $\Lambda[\lambda]$, we may assume $(x, y) \in \Lambda^{\oplus r+t}$. In particular, there exists an element $z \in \Lambda^{\oplus t}$ such that

$$x^t\Sigma x + dy = dz,$$

and it remains to show we can ensure $z = 0 \mod \varpi$. We do know $dz = 0 \mod \varpi$. So $dz \in \mathrm{span}(\varpi e_2, \dots, \varpi e_s)$. Say $dz = \varpi u$, we have $u \in \mathrm{span}(e_2, \dots, e_s) \subset \mathrm{Im}(d)$. Say $u = dv$. So $dz = \varpi dv = d\varpi v$. By replacing z by ϖv, we have

$$x^t\Sigma x + dy = d\varpi v.$$

Finally, replacing y by $(y - \varpi v)$, we are done. □

We will call affine varieties defined by Heisenberg equations *Heisenberg varieties*. Theorem 2.3 says all Λ/ϖ-points of a Heisenberg variety admit a ϖ-adic thickening.

3. Extensions of (φ, Γ)-modules and non-abelian (φ, Γ)-cohomology

Fix a pinned split reductive group $(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$ over $\mathbb{Z}$. Fix a parabolic $P \subset \widehat{G}$ containing $\widehat{B}$ with Levi subgroupn $\widehat{M}$ and unipotent radical U. Let K be a p-adic field. Let A be a $\mathbb{Z}_p$-algebra. The ring $\mathbb{A}_{K,A}$ is defined as in [L23B, Definition 4.2.8]. See [L23B, Section 4.2] for the definition of the procyclic group H_K. Fix a topological generator γ of H_K. Note that $\mathbb{A}_{K,A}$ admits a Frobenius action φ which commutes γ.

3.1. Framed parabolic (φ, Γ)-modules A *framed* (φ, γ)*-module* with P-structure and A-coefficients is a pair of matrices $[\phi], [\gamma] \in P(\mathbb{A}_{K,A})$, satisfying $[\phi]\varphi([\gamma]) = [\gamma]\gamma([\phi])$.

A *framed* (φ, Γ)*-module* with P-structure and A-coefficients is a framed (φ, γ)-module $([\phi], [\gamma])$ with P-structure and A-coefficients such that there exists a closed algebraic group embedding $P \hookrightarrow \mathrm{GL}_d = \mathrm{GL}(V)$ and $1 - [\gamma]$ induces a topologically nilpotent γ-semilinear endomorphism of $V(\mathbb{A}_{K,A})$. To make is concrete, if $\{e_1, \dots, e_d\}$ is the standard basis of V, then $[\gamma]$ sends αe_i to $\gamma(\alpha)[\gamma]e_i$.

3.2. Levi factor of (φ,γ)-modules Let $([\varphi],[\gamma])$ be a framed (φ,γ)-module with P-structure and A-coefficients. Write $([\varphi]_{\widehat{M}},[\gamma]_{\widehat{M}})$ for its image under the projection $P\to\widehat{M}$. Note that $([\varphi]_{\widehat{M}},[\gamma]_{\widehat{M}})$ is a framed (φ,γ)-module with $\widehat{M}$-structure.

3.3. Lemma Let $([\varphi],[\gamma])$ be a framed (φ,γ)-module with P-structure and A-coefficients. Then $([\varphi],[\gamma])$ is a framed (φ,Γ)-module with P-structure and A-coefficients if and only if $([\varphi]_{\widehat{M}},[\gamma]_{\widehat{M}})$ is a framed (φ,Γ)-module with $\widehat{M}$-structure and A-coefficients.

Proof. Note that $P=U\rtimes\widehat{M}$ and $\widehat{M}$ is a subgroup of P. We will regard both P and $\widehat{M}$ as a subgroup of $\mathrm{GL}_d\subset\mathrm{Mat}_{d\times d}$ by fixing an embedding $P\hookrightarrow\mathrm{GL}_d$.

Write $[u]:=[\gamma]-[\gamma]_{\widehat{M}}$. Note that the Jordan decomposition of $[\gamma]$ and $[\gamma]_{\widehat{M}}$ has the same semisimple part; write $[\gamma]=g_sg_u$ and $[\gamma]_{\widehat{M}}=g_sg'_u$ for the Jordan decomposition where both g_u and g'_u lies in the unipotent radical of a Borel of P (if we replace $\widehat{M}$ by one of its conjugate in P, then g_u and g'_u lies in the unipotent radical of the same Borel of P). By the Lie-Kolchin theorem, $(g_u-g'_u)$ is nilpotent. Since g_s commutes with g_u and g'_u, $[u]$ is nilpotent. We have $(1-[\gamma]_{\widehat{M}})=(1-[\gamma]+[u])$. Since $[u]$ is nilpotent, $(1-[\gamma]_{\widehat{M}})$ is topologically nilpotent if and only if $(1-[\gamma])$ is topologically nilpotent. □

3.4. Extensions of (φ,Γ)-modules In this paragraph, we classify all framed (φ,Γ)-modules with P-structure and A-coefficients whose Levi factor is equal to a fixed (φ,Γ)-module with $\widehat{M}$-structure $([\phi]_{\widehat{M}},[\gamma]_{\widehat{M}})$.

For ease of notation, write $f=[\phi]_{\widehat{M}}$ and $g=[\gamma]_{\widehat{M}}$. We denote by

$$H^1_{\mathrm{Herr}}(f,g)$$

the equivalence classes of all extensions of (f,g) to a framed (φ,Γ)-module with P-structure.

Let $u_f,u_g\in U(\mathbb{A}_{K,A})$. Set $[\phi]=u_ff$ and $[\gamma]=u_gg$. Note that $([\phi],[\gamma])$ is a (φ,Γ)-module if and only if

$$u_f\mathrm{Int}_g(\gamma(u_f^{-1}))=u_g\mathrm{Int}_f(\varphi(u_g^{-1}))$$

by Lemma 3.3.

Here $\mathrm{Int}_?(*)=?*?^{-1}$.

3.5. Assumption We assume U is a unipotent algebraic group of nilpotency class 2 and $p\neq 2$. Assume there exists an embedding $\iota:U\hookrightarrow\mathrm{GL}_N$ such that $(\iota(x)-1)^2=0$ for all $x\in U$.

In particular, there is a well-defined truncated log map

$$\begin{aligned}\log: U\to&\quad \mathrm{Lie}\,U\\ u\mapsto&\quad (u-1)-\tfrac{(u-1)^2}{2}\end{aligned}$$

whose inverse is the truncated exponential map

$$\exp:\mathrm{Lie}\,U\to U.$$

Here we embed U into $\mathrm{Mat}_{d\times d}$ in order to define addition and substraction (the embedding is not important). Write $x=\log(u_f)$ and $y=\log(u_g)$.

3.6. Lemma $([\phi],[\gamma])$ is a (φ,Γ)-module with P-structure extending $([\phi]_{\widehat{M}},[\gamma]_{\widehat{M}})$ if and only if

$$(1-\mathrm{Int}_g\circ\gamma)(x)-\frac{1}{2}[x,\mathrm{Int}_g\gamma(x)]=(1-\mathrm{Int}_f\circ\varphi)(y)-\frac{1}{2}[y,\mathrm{Int}_f\varphi(y)].$$

Proof. It follows from the Baker-Campbell-Hausdorff formula. □

Recall that a nilpotent Lie algebra of nilpotency class 2 is isomorphic to its associated graded Lie algebra (with respect to either the lower or the upper central filtration). We fix such an isomorphism $\mathrm{Lie}\,U\cong\mathrm{gr}^\bullet\mathrm{Lie}\,U=\mathrm{gr}^1\mathrm{Lie}\,U\oplus\mathrm{gr}^0\mathrm{Lie}\,U$ where $\mathrm{gr}^0\mathrm{Lie}\,U$ is the derived subalgebra of $\mathrm{Lie}\,U$. Note that $\mathrm{gr}^0\mathrm{Lie}\,U$ is contained in the center of $\mathrm{Lie}\,U$. In particular, if $x\in\mathrm{Lie}\,U$, we can write $x=x_0+x_1$ where $x_i\in\mathrm{gr}^i\mathrm{Lie}\,U$.

3.7. Lemma $([\phi],[\gamma])$ is a (φ,Γ)-module extending $([\phi]_{\widehat{M}},[\gamma]_{\widehat{M}})$ if and only if

$$\begin{cases}(1-\mathrm{Int}_g\circ\gamma)(x_1)-(1-\mathrm{Int}_f\circ\varphi)(y_1)=0\\(1-\mathrm{Int}_g\circ\gamma)(x_0)-(1-\mathrm{Int}_f\circ\varphi)(y_0)=\tfrac{1}{2}[x_1,\mathrm{Int}_g\gamma(x_1)]-\tfrac{1}{2}[y_1,\mathrm{Int}_f\varphi(y_1)].\end{cases}$$

Proof. Arrange terms according to their degree in the graded Lie algebra. □

3.8. Herr complexes Let V be a vector space over $\mathrm{Spec}\,A$, and let (s,t) be a framed (φ,Γ)-module with $\mathrm{GL}(V)$-structure and A-coefficients. Then the Herr complex associated to (s,t) is by definition the following

$$C^\bullet_{\mathrm{Herr}}(s,t):=[V(\mathbb{A}_{K,A})\xrightarrow{(s\circ\varphi-1,t\circ\gamma-1)}V(\mathbb{A}_{K,A})\oplus V(\mathbb{A}_{K,A})\xrightarrow{(t\circ\gamma-1,1-s\circ\varphi)^t}V(\mathbb{A}_{K,A})]$$

Write $Z^\bullet_{\mathrm{Herr}}(s,t)$, $B^\bullet_{\mathrm{Herr}}(s,t)$ and $H^\bullet_{\mathrm{Herr}}(s,t)$ for the cocycle group, the coboundary group and the cohomology group of $C^\bullet_{\mathrm{Herr}}(s,t)$. The reader can easily check that our definiton is consistent with that of [EG23, Section 5.1]. We will denote by d the differential operators in $C^\bullet_{\mathrm{Herr}}(f,g)$.

Note that $\widehat{M}$ acts on $\mathrm{gr}^1\mathrm{Lie}(U)$ and $\mathrm{gr}^0\mathrm{Lie}(U)$ by conjugation. Write

$$\mathrm{Int}^0:\widehat{M}\to\mathrm{GL}(\mathrm{gr}^0\mathrm{Lie}(U)),$$
$$\mathrm{Int}^1:\widehat{M}\to\mathrm{GL}(\mathrm{gr}^1\mathrm{Lie}(U))$$

for the conjugation actions.

3.9. Cup products Define a map

$$Q:C^1_{\mathrm{Herr}}(\mathrm{Int}^1(f,g))\to C^2_{\mathrm{Herr}}(\mathrm{Int}^0(f,g))$$
$$(x_1,y_1)\mapsto\frac{1}{2}[x_1,\mathrm{Int}^1_g\gamma(x_1)]-\frac{1}{2}[y_1,\mathrm{Int}^1_f\varphi(y_1)]$$

and a symmetric bilinear pairing

$$\cup:C^1_{\mathrm{Herr}}(\mathrm{Int}^1(f,g))\times C^1_{\mathrm{Herr}}(\mathrm{Int}^1(f,g))\to C^2_{\mathrm{Herr}}(\mathrm{Int}^0(f,g))$$
$$((x_1,y_1),(x_1',y_1'))\mapsto\frac{1}{2}(Q(x_1+x_1',y_1+y_1')-Q(x_1,y_1)-Q(x_1',y_1')).$$

3.10. Proposition Define

$$Z^1_{\mathrm{Herr}}(f,g) := \{(x_0+x_1, y_0+y_1) \in C^1_{\mathrm{Herr}}(\mathrm{Int}^0(f,g)\oplus \mathrm{Int}^1(f,g)) | \begin{cases} (x_0,y_0)\in C^1_{\mathrm{Herr}}(\mathrm{Int}^0(f,g)) \\ (x_1,y_1)\in C^1_{\mathrm{Herr}}(\mathrm{Int}^1(f,g)) \\ d(x_1,y_1)=0 \\ d(x_0,y_0)+(x_1,y_1)\cup(x_1,y_1)=0 \end{cases} \}.$$

There exists a surjective map

$$\begin{aligned} Z^1_{\mathrm{Herr}}(f,g) &\to H^1_{\mathrm{Herr}}(f,g) \\ (x,y) &\mapsto (\exp(x)f, \exp(y)g) \end{aligned}$$

Proof. It is a reformulation of Lemma 3.7. □

3.11. Lemma The cup product induces a well-defined symmetric bilinear pairing

$$\cup_H : H^1_{\mathrm{Herr}}(\mathrm{Int}^1(f,g)) \times H^1_{\mathrm{Herr}}(\mathrm{Int}^1(f,g)) \to H^2_{\mathrm{Herr}}(\mathrm{Int}^0(f,g)).$$

Proof. The proof is formally similar to [L21, Lemma 2.3.3.2]. □

3.12. Non-split groups We remark that all results in this section holds for non-split groups. More precisely, Let $F\subset K$ be a p-adic field and fix an action of $\Delta := \mathrm{Gal}(K/F)$ on the pinned group $(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$ and assume both $\widehat{M}$ and P are Δ-stable.

Set ${}^LG := \widehat{G}\rtimes\Delta$, ${}^LP := P\rtimes\Delta$, and ${}^LM\rtimes\Delta$. Denote by $\mathrm{GL}^!(\mathrm{Lie}\,U)$ the (parabolic) subgroup of the general linear group $\mathrm{GL}(\mathrm{Lie}\,U)$ that preserves the lower central filtration of $\mathrm{Lie}\,U$. Using the truncated log/exp map, we have a group scheme homomorphism ${}^LM\to\mathrm{GL}^!(\mathrm{Lie}\,U)$, which extends to a group scheme homomorphism

$${}^LP = {}^LM\rtimes U \to \mathrm{GL}^!(\mathrm{Lie}\,U)\rtimes U.$$

Name the group $\mathrm{GL}^!(\mathrm{Lie}\,U)\rtimes U$ as $\widetilde{P}$, and name the homomorphism ${}^LP\to\widetilde{P}$ as Ξ.

3.13. Definition In the non-split setting, a framed (φ,Γ)-module with ${}^LP$-structure is a (φ,Γ)-module (F,ϕ_F,γ_F) with ${}^LP$-structure, and a framed (φ,Γ)-module $([\phi],[\gamma])$ with $\widetilde{P}$-structure, together with an identification $\Xi_*(F,\phi_F,\gamma_F)\cong([\phi],[\gamma])$.

The reason we make the definition above is because (φ,Γ)-modules with H-structure are not represented by a pair of matrices if H is a *disconnected* group. So by choosing the map ${}^LP\to\widetilde{P}$, we are able to work with connected groups $\widetilde{P}$.

Since the whole purpose of this section is to understand extensions of (φ,Γ)-modules and we fix the ${}^LM$-semisimplification of framed (φ,Γ)-module with ${}^LP$-structure, the reader can easily see all results carry over to the non-split case by using $\widetilde{P}$ in place of P.

4. Cohomologically Heisenberg lifting problems

We keep notations from the previous section. Let $\Lambda \subset \bar{\mathbb{Z}}_p$ be a DVR. Let $([\bar{\phi}], [\bar{\gamma}])$ be a framed (φ, Γ)-module with P-structure and $\bar{\mathbb{F}}_p$-coefficients. Write $(\bar{f}, \bar{g})$ for $([\bar{\phi}]_{\widehat{M}}, [\bar{\gamma}]_{\widehat{M}})$. Fix a framed (φ, Γ)-module (f, g) with $\widehat{M}$-structure and Λ-coefficients lifting $(\bar{f}, \bar{g})$.

By Proposition 3.10, there exists an element $(\bar{x}, \bar{y}) \in Z^1_{\mathrm{Herr}}(\bar{f}, \bar{g})$ representing $([\bar{\phi}], [\bar{\gamma}])$. We can write $\bar{x} = \bar{x}_0 + \bar{x}_1$ and $\bar{y} = \bar{y}_0 + \bar{y}_1$ such that $(\bar{x}_i, \bar{y}_i) \in C^1_{\mathrm{Herr}}(\mathrm{Int}^i(\bar{f}, \bar{g}))$.

4.1. Definition A *cohomologically Heisenberg lifting problem* is a tuple $(f, g, \bar{x}, \bar{y}, H)$ consisting of

- a framed (φ, Γ)-module with $\widehat{M}$-structure and Λ-coefficients (f, g);
- an element $(\bar{x}, \bar{y}) \in Z^1_{\mathrm{Herr}}(\bar{f}, \bar{g})$, and
- a Λ-submodule $H \subset H^1_{\mathrm{Herr}}(\mathrm{Int}^1(f, g))$,

such that

(HL1) $(\bar{x}_1, \bar{y}_1)$ lies in the image of H in $H^1_{\mathrm{Herr}}(\mathrm{Int}^1(\bar{f}, \bar{g}))$,
(HL2) the pairing $\cup_H|_H : H \times H \to H^2_{\mathrm{Herr}}(\mathrm{Int}^0(f, g))$ is surjective,
(HL3) $H^2_{\mathrm{Herr}}(\mathrm{Int}^0(\bar{f}, \bar{g})) \cong \Lambda/\varpi$.

A *solution* to the lifting problem $(f, g, \bar{x}, \bar{y}, H)$ is an element $(x, y) \in Z^1_{\mathrm{Herr}}(f, g)$ lifting $(\bar{x}, \bar{y})$ such that the image of (x_1, y_1) in $H^1_{\mathrm{Herr}}(\mathrm{Int}^1(f, g))$ is contained in H.

At first sight, a cohomologically Heisenberg lifting problem defines an infinite system of quadratic polynomial equations. In the following theorem, we show that we may truncate the infinite system to a finite system and solve cohomologically Heisenberg lifting problems.

4.2. Theorem Each cohomologically Heisenberg lifting problem is solvable after replacing Λ by an extension of DVR $\Lambda' \subset \bar{\mathbb{Z}}_p$.

Proof. Before we start, we remark that $C^\bullet_{\mathrm{Herr}}(\mathrm{Int}^i(f, g)) \otimes_\Lambda \Lambda/\varpi = C^\bullet_{\mathrm{Herr}}(\mathrm{Int}^i(\bar{f}, \bar{g}))$ while it is *not* generally true that $H^\bullet_{\mathrm{Herr}}(\mathrm{Int}^i(f, g)) \otimes_\Lambda \Lambda/\varpi = H^\bullet_{\mathrm{Herr}}(\mathrm{Int}^i(\bar{f}, \bar{g}))$.

Write Z_H for the preimage of H in $Z^1_{\mathrm{Herr}}(\mathrm{Int}^1(f, g))$. Since $Z^1_{\mathrm{Herr}}(\mathrm{Int}^1(f, g))$ is Λ-torsion-free, so is Z_H. Let $X \subset Z_H$ be a finite Λ-submodule which maps surjectively onto H. Such an X exists because $H^1_{\mathrm{Herr}}(\mathrm{Int}^1(f, g))$ is a finite Λ-module ([EG23, Theorem 5.1.22]). Since Λ is a DVR, X is finite free over Λ.

Let $W \subset C^2_{\mathrm{Herr}}(\mathrm{Int}^0(f, g))$ be a finite free Λ-submodule containing $X \cup X$.

By (HL2), $W \to H^2_{\mathrm{Herr}}(\mathrm{Int}^0(f, g))$ is surjective. Set $B_W := B^2_{\mathrm{Herr}}(\mathrm{Int}^0(f, g)) \cap W$, we have $H^2_{\mathrm{Herr}}(\mathrm{Int}^0(f, g)) = W/B_W$. Again, B_W is a finite free Λ-module.

Finally, let $Y \subset C^1_{\mathrm{Herr}}(\mathrm{Int}^0(f, g))$ be a finite free Λ-submodule which maps surjectively onto B_W and contains at least one lift of $(\bar{x}_0, \bar{y}_0)$.

Now consider the system of equations

$$(\dagger) \qquad \mathfrak{x} \cup \mathfrak{x} + d\mathfrak{y} = 0 \in W$$

where $\mathfrak{x} \in X$ and $\mathfrak{y} \in Y$ are the variables and W is the value space. We check that $(\dagger)$ is a Heisenberg equation in the sense of Definition 2.2. (H1) follows from (HL3) and the Nakayama lemma, while (H2) follows from (HL2). The equation $(\dagger)$ admits a mod ϖ solution $\bar{\mathfrak{x}}, \bar{\mathfrak{y}}$ defined by $(\bar{x}, \bar{y}) \in Z^1_{\mathrm{Herr}}(\bar{f}, \bar{g})$. By

Theorem 2.3, (†) admits a solution lifting $\bar{\mathfrak{x}}, \bar{\mathfrak{y}}$ after extending the coefficient ring Λ. The solution to the equation (†) is also a solution to the lifting problem. □

5. Applications to Galois cohomology

Let $F/\mathbb{Q}_p$ be a p-adic field, and let G is a tamely ramified quasi-split reductive group over F which splits over K. Write $\Delta := \mathrm{Gal}(K/F)$. There exists a Δ-stable pinning $(G, B, T, \{X_\alpha\})$ of G, and let $(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$ be the dual pinned group. Let $P \subset \widehat{G}$ be a Δ-stable parabolic of $\widehat{G}$ with Δ-stable Levi subgroup $\widehat{M}$ and unipotent radical U. Denote by ${}^LP$ the semi-direct product $U \rtimes {}^LM$ where ${}^LM = \widehat{M} \rtimes \Delta$.

In the terminology of [L21], ${}^LP$ is a big pseudo-parabolic of ${}^LG = \widehat{G} \rtimes \Delta$ and all big pseudo-parabolic of ${}^LG$ are of the form ${}^LP$ (see [L21, Section 3]).

We enforce Assumption 3.5 throughout this section. Note that ${}^LM$ acts on $\mathrm{Lie}\, U = \mathrm{gr}^0 \mathrm{Lie}\, U \oplus \mathrm{gr}^1 \mathrm{Lie}\, U$ by adjoint, and we denote the adjoint actions by Int^i as in Paragraph 3.8.

5.1. Definition Let $\bar{\rho}_P : \mathrm{Gal}_F \to {}^LP(\bar{\mathbb{F}}_p)$ be a Langlands parameter with Levi factor $\bar{\rho}_M : \mathrm{Gal}_F \to {}^LM(\bar{\mathbb{F}}_p)$. We say $\bar{\rho}_P$ is a Heisenberg-type extension of $\bar{\rho}_M$ if

$$\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \mathrm{Lie}\, U(\bar{\mathbb{F}}_p)) \leq 1.$$

Here the Gal_F-action on $\mathrm{gr}^0 \mathrm{Lie}\, U(\bar{\mathbb{F}}_p)$ is obtained from composing $\bar{\rho}_M$ and $\mathrm{Int}^0 : {}^LM \to \mathrm{GL}(\mathrm{gr}^0 \mathrm{Lie}\, U)$.

5.2. Cup products on Galois cohomology Let A be either $\bar{\mathbb{F}}_p$ or $\bar{\mathbb{Z}}_p$. If $\rho_M : \mathrm{Gal}_F \to {}^LM(A)$ is an L-parameter, we can equip $\mathrm{Lie}\, U(A)$ with Gal_F-action via ρ_M. Note that there exists a symmetric bilinear pairing

$$H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \mathrm{Lie}\, U(A)) \times H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \mathrm{Lie}\, U(A)) \to H^2(\mathrm{Gal}_F, \mathrm{gr}^0 \mathrm{Lie}\, U(A)),$$

which is defined in [L21, Section 3.2]. Alternatively, we can transport the symmetric cup product on (φ, Γ)-cohomology defined in Definition 3.9 and Lemma 3.11, and later generalized in 3.12 to Galois cohomology.

5.3. Partial extensions and partial lifts A *partial extension of* ρ_M is a continuous group homomorphism $\rho' : \mathrm{Gal}_F \to \frac{{}^LP}{[U,U]}(\bar{\mathbb{Z}}_p)$ extending $\rho_M : \mathrm{Gal}_F \to {}^LM(\bar{\mathbb{Z}}_p) = \frac{{}^LP}{U}(\bar{\mathbb{Z}}_p)$. Here $[U,U]$ is the derived subgroup of U. The set of equivalence classes of partial extensions of ρ_M are in natural bijection with $H^1(\mathrm{Gal}_F, \frac{U}{[U,U]}(\bar{\mathbb{Z}}_p)) = H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \mathrm{Lie}\, U(\bar{\mathbb{Z}}_p))$.

Let $\bar{\rho}_P : \mathrm{Gal}_F \to {}^LP(\bar{\mathbb{F}}_p)$ be an L-parameter. A *partial lift* of $\bar{\rho}_P$ is a group homomorphism $\rho' : \mathrm{Gal}_F \to \frac{{}^LP}{[U,U]}(\bar{\mathbb{Z}}_p)$ which lifts $\bar{\rho}_P$ mod $[U,U]$.

5.4. Lemma A partial extension $\rho' : \mathrm{Gal}_F \to \frac{{}^LP}{[U,U]}(A)$ of $\rho_{M,A}$ extends to a full extension $\rho : \mathrm{Gal}_F \to {}^LP(A)$ if and only if $c \cup c = 0$ where $c \in H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \mathrm{Lie}\, U(A))$ is the cohomology class corresponding to ρ'.

Proof. It follows immediately from Proposition 3.10. □

5.5. Theorem Assume $p \neq 2$. Let $\bar{\rho}_P : \mathrm{Gal}_F \to {}^L P(\bar{\mathbb{F}}_p)$ be an extension of $\bar{\rho}_M$. Assume $\bar{\rho}_M$ admits a lift $\rho_M : \mathrm{Gal}_F \to {}^L M(\bar{\mathbb{Z}}_p)$ such that

(i) $\bar{\rho}_P$ is a Heisenberg-type extension of $\bar{\rho}_M$,
(ii) $\bar{\rho}_P|_{\mathrm{Gal}_K}$ admits a partial lift which is a partial extension of $\rho_M|_{\mathrm{Gal}_K}$, and
(iii) the pairing

$$H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_F, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p))$$

is non-trivial unless $H^2(\mathrm{Gal}_F, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p)) = 0$,

then $\bar{\rho}_P$ admits a lift $\rho_P : \mathrm{Gal}_F \to {}^L P(\bar{\mathbb{Z}}_p)$ with Levi factor ρ_M.

Proof. Let A be either $\bar{\mathbb{F}}_p$ or $\bar{\mathbb{Z}}_p$, and let $\rho_{M,A}$ be either $\bar{\rho}_M$ or ρ_M, resp.. The set of equivalence classes of L-parameters $\mathrm{Gal}_F \to {}^L P/[U,U](A)$ extending $\rho_{M,A}$ is in natural bijection with the A-module $H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \operatorname{Lie} U(A))$. Since K/F is assumed to have prime-to-p degree, we have

$$H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \operatorname{Lie} U(A)) = H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A))^{\Delta}$$

by [Ko02, Theorem 3.15].

The L-parameter $\bar{\rho}_P$ defines an element $\bar{c} \in H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{F}}_p))^{\Delta}$. By item (ii), there exists a lift $c' \in H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))$ lifting $\bar{c}$. Define $c := \frac{1}{[K:F]} \sum_{\gamma \in \Delta} \gamma c \in H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))$. It is clear that c lifts $\bar{c}$.

There are two possibilities: either

$$\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_F, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p)) = 0,$$

or

$$\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_F, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p)) = 1.$$

In the former case, there is no obstruction to extension and lifting and the Theorem follows from [L21, Proposition 5.3.1]. Now we consider the latter case.

By Fontaine's theory of (φ, Γ)-modules, ρ_M corresponds to a framed (φ, Γ)-module (f, g) with ${}^L M$-structure (or rather $\mathrm{Gal}^!(\operatorname{Lie} U)$-structure by Paragraph 3.12), and $\bar{\rho}_P$ corresponds to an element $(\bar{x}, \bar{y}) \in Z^1_{\mathrm{Herr}}(\bar{f}, \bar{g})$. Here $(\bar{f}, \bar{g})$ is the reduction of (f, g). Since Galois cohomology is naturally isomorphic to the cohomology of Herr complexes ([EG23, Theorem 5.1.29]), we can identify $H^1_{\mathrm{Herr}}(\mathrm{Int}^i(f, g))$ with $H^1(\mathrm{Gal}_F, \mathrm{gr}^i \operatorname{Lie} U(\bar{\mathbb{Z}}_p))$.

Consider the tuple $(f, g, \bar{x}, \bar{y}, H^1(\mathrm{Gal}_F, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)))$. We want to check that this tuple is a cohomologically Heisenberg lifting problem in the sense of Definition 4.1. (HL3) follows from assumption (i), (HL2) follows from assumption (iii) and (HL3), and (HL1) follows from assumption (ii) and the discussion in the second paragraph of this proof. We finish the proof by invoking Theorem 4.2. ∎

6. Interaction of cup products with $\mathbb{Z}/2$-action

Let K be a p-adic field. Let $a = c$ and b be positive integers. Fix a Galois representation

$$\bar{\tau} = \begin{bmatrix} \bar{\tau}_a & & \\ & \bar{\tau}_b & \\ & & \bar{\tau}_c \end{bmatrix} : \mathrm{Gal}_K \to \begin{bmatrix} \mathrm{GL}_a & & \\ & \mathrm{GL}_b & \\ & & \mathrm{GL}_c \end{bmatrix} (\bar{\mathbb{F}}_p),$$

as well as a lift

$$\tau = \begin{bmatrix} \tau_a & & \\ & \tau_b & \\ & & \tau_c \end{bmatrix} : \mathrm{Gal}_K \to \begin{bmatrix} \mathrm{GL}_a & & \\ & \mathrm{GL}_b & \\ & & \mathrm{GL}_c \end{bmatrix} (\bar{\mathbb{Z}}_p)$$

of $\bar{\tau}$. Write $\mathrm{gr}^0 \operatorname{Lie} U := \mathrm{Mat}_{a\times c}$, and $\mathrm{gr}^1 \operatorname{Lie} U := \mathrm{Mat}_{a\times b} \oplus \mathrm{Mat}_{b\times c}$. Recall that we have defined a (symmetrized) cup product

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A)) \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A)) \to H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(A))$$

for $A = \bar{\mathbb{F}}_p, \bar{\mathbb{Z}}_p$. If $c \in H^i(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A))$, write $c = (c_1, c_2)$ where $c_1 \in H^i(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(A))$, and $c_2 \in H^i(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(A))$,

6.1. Lemma For the cup product

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A)) \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A)) \to H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(A)),$$

we have

$$\begin{aligned} (c_1, 0) \cup (c_1, 0) &= 0 \\ (0, c_2) \cup (0, c_2) &= 0 \\ (c_1, 0) \cup (0, c_2) &= \frac{1}{2}(c_1, c_2) \cup (c_1, c_2) \\ (c_1, 0) \cup (c_1', 0) &= 0 \\ (0, c_2) \cup (0, c_2') &= 0, \end{aligned}$$

for any c_1, c_2, c_1', c_2'.

Proof. The first two identities follows from Lemma 5.4. The last three identity follows from the first two identities. ☐

Write Δ for the finite group $\{1, \jmath\}$ with two elements. While Δ denotes the Galois group $\mathrm{Gal}(K/F)$ in the previous sections, Δ is merely an abstract group in this section.

6.2. Definition An action of Δ on each of $H^i(\mathrm{Gal}_K, \mathrm{gr}^j \operatorname{Lie} U(A))$ is said to be *classical* if

- $\jmath H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(A)) \subset H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(A))$,
- $\jmath H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(A)) \subset H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(A))$,
- the Δ-action is compatible with the cup products.

We also call such Δ-actions *classical Δ-structure*.

6.3. Proposition Fix a classical Δ-structure. If

$$H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p)) = H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p))^\Delta = \bar{\mathbb{F}}_p,$$

then the cup product

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^\Delta \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^\Delta \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_F, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p))^\Delta$$

is non-trivial if and only if

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p))$$

is non-trivial.

Proof. Since $\cup$ is a symmetric pairing, non-triviality of $\cup$ on $H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))$ implies $(c_1, c_2) \cup (c_1, c_2) \neq 0$ for some $(c_1, c_2) \in H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p$. By Lemma 6.1, $(c_1, 0) \cup (0, c_2) \neq 0$. Since

$$H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p))^{\Delta} = H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p)),$$

we conclude Δ-acts trivially on $H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \operatorname{Lie} U(\bar{\mathbb{F}}_p))$.

We argue by contradition and assume $\cup$ is trivial on $H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p$. We claim $x \cup y = 0$ for each $x \in H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p$ and $y \in H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p$. Indeed,

$$2(x \cup y) = x \cup y + \jmath(x \cup y) = x \cup y + (\jmath x) \cup (\jmath y) = x \cup y + x \cup \jmath y = x \cup (y + \jmath y) = 0$$

because both x and $y + \jmath y$ lies in $H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p$.

Since $(c_1, 0) + \jmath(c_1, 0) \in H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p$, we have $((c_1, 0) + \jmath(c_1, 0)) \cup (0, c_2) = 0$. However, since $(c_1, 0) \cup (0, c_2) \neq 0$, we must have $\jmath(c_1, 0) \cup (0, c_2) \neq 0$. By the classicality of the Δ-structure, $\jmath(c_1, 0) = (0, c_2')$ for some c_2' and thus by Lemma 6.1, $\jmath(c_1, 0) \cup (0, c_2) = 0$ and we get a contradiction. □

Next, we establish a general non-triviality of cup products result, and before that we need a non-degeneracy result.

6.4. Lemma Assume $\bar{\tau}_a$ and $\bar{\tau}_c$ are irreducible. Then the cup product

$$H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p)) \times H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(\bar{\mathbb{F}}_p)) \to H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times c}(\bar{\mathbb{F}}_p))$$

is non-degenerate.

Proof. Fix a non-zero element $x \in H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(\bar{\mathbb{F}}_p))$. Such an extension class x corresponds to a non-split extension $\bar{\tau}_d = \begin{bmatrix} \bar{\tau}_b & * \\ & \bar{\tau}_c \end{bmatrix}$. In particular, the map

$$H^0(\mathrm{Gal}_K, \bar{\tau}_a^{\vee} \otimes \bar{\tau}_d(1)) \to H^0(\mathrm{Gal}_K, \bar{\tau}_a^{\vee} \otimes \bar{\tau}_c(1))$$

is the zero map (if otherwise the socle of $\bar{\tau}_d$ is strictly larger than the socle of $\bar{\tau}_b$ and $\bar{\tau}_d$ must be a split extension). By local Tate duality, the map

$$H^2(\mathrm{Gal}_K, \bar{\tau}_c^{\vee} \otimes \bar{\tau}_a) \to H^2(\mathrm{Gal}_K, \bar{\tau}_d^{\vee} \otimes \bar{\tau}_a)$$

is also the zero map. The short exact sequence

$$0 \to \bar{\tau}_b \to \bar{\tau}_d \to \bar{\tau}_c \to 0$$

induces the long exact sequence

$$H^1(\mathrm{Gal}_K, \bar{\tau}_c^{\vee} \otimes \bar{\tau}_a) \to H^1(\mathrm{Gal}_K, \bar{\tau}_d^{\vee} \otimes \bar{\tau}_a) \to H^1(\mathrm{Gal}_K, \bar{\tau}_b^{\vee} \otimes \bar{\tau}_a) \to H^2(\mathrm{Gal}_K, \bar{\tau}_c^{\vee} \otimes \bar{\tau}_a) \xrightarrow{0} H^2(\mathrm{Gal}_K, \bar{\tau}_d^{\vee} \otimes \bar{\tau}_a).$$

Since $H^2(\mathrm{Gal}_K, \bar{\tau}_c^{\vee} \otimes \bar{\tau}_a) \neq 0$, there exists an element $y \in H^1(\mathrm{Gal}_K, \bar{\tau}_b^{\vee} \otimes \bar{\tau}_a)$ which maps to a non-zero element of $H^2(\mathrm{Gal}_K, \bar{\tau}_c^{\vee} \otimes \bar{\tau}_a)$, and y does not admit an extension to $H^1(\mathrm{Gal}_K, \bar{\tau}_d^{\vee} \otimes \bar{\tau}_a)$. By Lemma 5.4, $(x, y) \cup (x, y) \neq 0$, and thus by Lemma 6.1, $x \cup y = \frac{1}{2}((x, y) \cup (x, y)) \neq 0$. □

6.5. Lemma Let X, Y be vector spaces over a field κ. Let

$$\cup : X \times Y \to \kappa$$

be a non-degenerate bilinear pairing. Let $H_X \subset X$ and $H_Y \subset Y$ be subspaces such that $x \cup y = 0$ for all $x \in H_X$ and $y \in H_Y$. Then either $\dim X \geq 2 \dim H_X$ or $\dim Y \geq 2 \dim H_Y$.

Proof. It suffices to show $\dim X + \dim Y \geq 2(\dim H_X + \dim H_Y)$. We define a (symmetric) inner product structure on $X \oplus Y$ by setting $(x, y) \cdot (x, y) = x \cup y + y \cup x$. The lemma now follows from the Gram-Schmidt process. $\square$

6.6. Lemma Assume both $\bar{\tau}_a$ and $\bar{\tau}_c$ are irreducible. The cup product

$$H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times c}(\bar{\mathbb{F}}_p))$$

is non-trivial unless all of the following holds

- $a = 1$,
- $K = \mathbb{Q}_p$,
- either $\bar{\tau}_b = \bar{\tau}_a(-1)^{\oplus b}$ and $\bar{\tau}_c = \bar{\tau}_a(-1)$; or $\bar{\tau}_b = \bar{\tau}_a^{\oplus b}$ and $\bar{\tau}_c = \bar{\tau}_a(-1)$.

Proof. By Lemma 6.5 and Lemma 6.4, the lemma holds if

$$(*) \qquad \dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p > \frac{1}{2} \dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p))$$

and

$$(**) \qquad \dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p > \frac{1}{2} \dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b\times c}(\bar{\mathbb{F}}_p)).$$

We prove by contradiction and assume either $(*)$ or $(**)$ fails. Since $(*)$ and $(**)$ are completely similar, we assume the contraposition of $(*)$ that

$$(1) \qquad H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \leq \frac{1}{2} \dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p)).$$

By the universal coefficient theorem, we have the short exact sequence

$$0 \to H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p)) \to \mathrm{Tor}_1^{\bar{\mathbb{Z}}_p}(H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)), \bar{\mathbb{F}}_p) \to 0.$$

Therefore the assumption (1) is equivalent to

$$(2) \qquad \dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \leq \dim_{\bar{\mathbb{F}}_p} \mathrm{Tor}_1^{\bar{\mathbb{Z}}_p}(H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)), \bar{\mathbb{F}}_p).$$

Note that

$$\dim_{\bar{\mathbb{F}}_p} \mathrm{Tor}_1^{\bar{\mathbb{Z}}_p}(H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)), \bar{\mathbb{F}}_p) \leq \dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p)),$$

since H^2 commutes with base change; also see [We94, Example 3.1.7] for the computation of Tor. Also note that

$$\dim_{\bar{\mathbb{F}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p)) \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \geq \mathrm{rank}_{\bar{\mathbb{Z}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p))_{\text{torsion-free}},$$

and

$$\mathrm{rank}_{\bar{\mathbb{Z}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p))_{\text{torsion-free}} = \dim_{\bar{\mathbb{Q}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Q}}_p)).$$

By local Euler characteristic, we have

$$\begin{aligned}\dim_{\bar{\mathbb{Q}}_p} H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Q}}_p)) =& \dim_{\bar{\mathbb{Q}}_p} H^0(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Q}}_p)) \\ &+ \dim_{\bar{\mathbb{Q}}_p} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Q}}_p)) + [K:\mathbb{Q}_p]ab \\ \geq& [K:\mathbb{Q}_p]ab.\end{aligned}$$

On the other hand, by local Tate duality

$$\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p)) = \dim_{\bar{\mathbb{F}}_p} H^0(\mathrm{Gal}_K, \tau_a^\vee \otimes \tau_b(1)) \leq ab.$$

Combine all above, (2) becomes

$$[K:\mathbb{Q}_p]ab \leq ab.$$

So, all inequalities above must be equalities and we are forced to have

(i) $K = \mathbb{Q}_p$,
(ii) $H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{Z}}_p))$ is torsion-free, and
(iii) $\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times b}(\bar{\mathbb{F}}_p)) = ab$.

Item (iii) further forces $a = 1$ because $\bar{\tau}_a$ is assumed to be irreducible. □

6.7. Corollary Fix a classical Δ-structure. Assume both $\bar{\tau}_a$ and $\bar{\tau}_c$ are irreducible. The cup product

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \mathrm{Lie}\, U(\bar{\mathbb{Z}}_p))^\Delta \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \mathrm{Lie}\, U(\bar{\mathbb{Z}}_p))^\Delta \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times c}(\bar{\mathbb{F}}_p))^\Delta$$

is non-trivial unless all of the following holds

- $a = 1$,
- $K = \mathbb{Q}_p$,
- either $\bar{\tau}_b = \bar{\tau}_a(-1)^{\oplus b}$ and $\bar{\tau}_c = \bar{\tau}_a(-1)$; or $\bar{\tau}_b = \bar{\tau}_a^{\oplus b}$ and $\bar{\tau}_c = \bar{\tau}_a(-1)$.

Proof. Combine Lemma 6.6 and Proposition 6.3. □

7. Example A: unitary groups

Now assume $G = U_n$ is a quasi-split tamely ramified unitary group over F which splits over the quadratic extension K/F (thus we have implicitly assumed $p \neq 2$).

The Dynkin diagram of G is a chain of $(n-1)$-vertices (•–•⋯•–•), and $\Delta = \mathrm{Gal}(K/F)$ acts on $\mathrm{Dyn}(G)$ by reflection. The maximal proper Δ-stable subsets of $\mathrm{Dyn}(G)$ are given by removing either two symmetric vertices, or the middle vertex. Therefore, the Levi subgroups of maximal proper F-parabolics of G are of the form

$$M_k := \mathrm{Res}_{K/F} \mathrm{GL}_k \times U_{n-2k}.$$

If ${}^L P$ is a maximal proper parabolic of ${}^L G$, then the Levi of ${}^L P$ is of the form ${}^L M_k$; we will write ${}^L P_k$ for ${}^L P$ to emphasize its type.

7.1. Proposition Let $\bar{\rho} : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ be an L-parameter. Then either $\bar{\rho}$ is elliptic, or $\bar{\rho}$ factors through ${}^L P_k(\bar{\mathbb{F}}_p)$ for some k such that the composite $\bar{r} : \mathrm{Gal}_F \xrightarrow{\bar{\rho}^{ss}} {}^L M_k(\bar{\mathbb{F}}_p) \to {}^L \mathrm{Res}_{K/F} \mathrm{GL}_k(\bar{\mathbb{F}}_p)$ is elliptic.

Proof. By [L23, Theorem B], $\bar{\rho}$ is either elliptic, or factors through some ${}^L P_k(\bar{\mathbb{F}}_p)$. By the non-abelian Shapiro's lemma, L-parameters $\mathrm{Gal}_F \to {}^L \mathrm{Res}_{K/F} \mathrm{GL}_k(\bar{\mathbb{F}}_p)$ are in natural bijection to L-parameters $\mathrm{Gal}_K \to \mathrm{GL}_k(\bar{\mathbb{F}}_p)$; and this bijection clearly preserves ellipticity. Suppose $\bar{r}$ is not elliptic, then $\bar{r}$, when regarded as a Galois representation $\mathrm{Gal}_K \to \mathrm{GL}_k(\bar{\mathbb{F}}_p)$, contains a proper irreducible subrepresentation $\bar{r}_0 : \mathrm{Gal}_K \to \mathrm{GL}_s(\bar{\mathbb{F}}_p)$. It is easy to see that $\bar{\rho}$ also factors through ${}^L P_s(\bar{\mathbb{F}}_p)$. So we are done. ☐

We take a closer look at ${}^L P_k$:

$$ {}^L P_k = \begin{bmatrix} \mathrm{GL}_k & \mathrm{Mat}_{k\times(n-2k)} & \mathrm{Mat}_{k\times k} \\ & \mathrm{GL}_{n-2k} & \mathrm{Mat}_{(n-2k)\times k} \\ & & \mathrm{GL}_k \end{bmatrix} \rtimes \Delta $$

7.2. Lemma If $\bar{\rho} : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ is not elliptic, then there exists a parabolic ${}^L P_k$ through which $\bar{\rho}$ factors and $\bar{\rho}$ is a Heisenberg-type extension of some $\bar{\rho}_M : \mathrm{Gal}_F \to {}^L M_k(\bar{\mathbb{F}}_p)$.

Proof. By Proposition 7.1, there exists a parabolic ${}^L P_k$ such that $\bar{r} : \mathrm{Gal}_F \xrightarrow{\bar{\rho}^{ss}} {}^L M_k(\bar{\mathbb{F}}_p) \to {}^L \mathrm{Res}_{K/F} \mathrm{GL}_k(\bar{\mathbb{F}}_p)$ is elliptic. Write

$$ \bar{r}|_{\mathrm{Gal}_K} = \begin{bmatrix} \bar{r}_1 & & \\ & 1_{n-2k} & \\ & & \bar{r}_2 \end{bmatrix}, $$

where $\bar{r}_1, \bar{r}_2 : \mathrm{Gal}_K \to \mathrm{GL}_k(\bar{\mathbb{F}}_p)$. By the non-abelian Shapiro's lemma (see [GHS, Subsection 9.4] for details), $\bar{r}$ can be fully reconstructed from $\bar{r}_1$, and $\bar{r}_2$ is completely determined by $\bar{r}_1$; in particular, both $\bar{r}_1$ and $\bar{r}_2$ are irreducible Galois representations. We have

$$ H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \mathrm{Lie}\, U(\bar{\mathbb{F}}_p)) = H^2(\mathrm{Gal}_K, \mathrm{Hom}(\bar{r}_2, \bar{r}_1)). $$

Since both $\bar{r}_1$ and $\bar{r}_2$ are irreducible, by local Tate duality, we have $\dim H^2(\mathrm{Gal}_K, \mathrm{Hom}(\bar{r}_2, \bar{r}_1)) = \dim H^0(\mathrm{Gal}_K, \mathrm{Hom}(\bar{r}_1, \bar{r}_2(1))) \le 1$. ☐

Next, we study cup products. Now fix the parabolic type ${}^L P_k$. We have

$$ \mathrm{gr}^1 \mathrm{Lie}\, U = \mathrm{Mat}_{k\times(n-2k)} \oplus \mathrm{Mat}_{(n-2k)\times k} $$

and

$$ \mathrm{gr}^0 \mathrm{Lie}\, U = \mathrm{Mat}_{k\times k}. $$

We will use all notations introduced in Section 6.

By [Ko02, Theorem 3.15], we have

$$ H^i(\mathrm{Gal}_F, \mathrm{gr}^j \mathrm{Lie}\, U(A)) = H^i(\mathrm{Gal}_K, \mathrm{gr}^j \mathrm{Lie}\, U(A))^{\mathrm{Gal}(K/F)} = H^i(\mathrm{Gal}_K, \mathrm{gr}^j \mathrm{Lie}\, U(A))^{\Delta}, $$

for all i and j.

7.3. Lemma The $\Delta = \mathrm{Gal}(K/F)$-action on $H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A))$ satisfies

$$\jmath(c_1, 0) = (0, *)$$
$$\jmath(0, c_2) = (*, 0),$$

for any c_1, c_2.

Proof. Write

$$w = \begin{bmatrix} 0 & 0 & J_1 \\ 0 & J_2 & 0 \\ J_3 & 0 & 0 \end{bmatrix}$$

for (a representative of) the longest Weyl group element. Let

$$\rho = \begin{bmatrix} A & B & * \\ & D & E \\ & & F \end{bmatrix} : \mathrm{Gal}_K \to P(A)$$

be a group homomorphism. Note that each of A, B, D, E, F is a matrix-valued function on Gal_K. Write A' for $\gamma \mapsto A(\jmath^{-1}\gamma\jmath)$ and similarly define B', D', E', F'. We have

$$\jmath\rho(\jmath^{-1} - \jmath)\jmath^{-1} = w\rho(\jmath^{-1} - \jmath)^{-t}w^{-1} = \begin{bmatrix} J_1 F'^{-t} J_1^{-1} & -J_1 F'^{-t} E'^{t} D'^{-t} J_2^{-1} & * \\ & J_2 B'^{-t} J_2^{-1} & -J_2 D'^{-t} B'^{t} A'^{-t} J_3^{-1} \\ & & J_3 A'^{-t} A^{-1} J_3^{-1} \end{bmatrix}.$$

In particular, we see that the $\widehat{\jmath}$-involution on $H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(A))$ permutes the two direct summands. ☐

The lemma above immediately implies the following.

7.4. Corollary The Galois action $\mathrm{Gal}(K/F)$ on $H^i(\mathrm{Gal}_K, \mathrm{gr}^j \operatorname{Lie} U(A))$ is a classical Δ-system in the sense of Definition 6.2.

7.5. Theorem Theorem 3 holds for ${}^L G_n = {}^L U_n$. Moreover, for unramified unitary groups, $\bar\rho$ admits a crystalline lift.

Proof. If $\bar\rho$ is elliptic, then it is [L23, Theorem C]. Suppose $\bar\rho$ is not elliptic, then by Lemma 7.2, there exists a parabolic ${}^L P_k$ through which $\bar\rho$ factors through and that $\bar\rho$ is a Heisenberg-type extension of some $\bar\rho_M : \mathrm{Gal}_F \to {}^L M_k(\bar{\mathbb{F}}_p)$. We have

$${}^L M_k = {}^L(\operatorname{Res}_{K/F} \mathrm{GL}_k \times U_{n-2k}) = \begin{bmatrix} \mathrm{GL}_k & & \\ & \mathrm{GL}_{n-2k} & \\ & & \mathrm{GL}_k \end{bmatrix} \rtimes \{1, \jmath\}$$

where

$$\begin{bmatrix} \mathrm{GL}_k & & \\ & I_{n-2k} & \\ & & \mathrm{GL}_k \end{bmatrix} \rtimes \{1, \jmath\} \cong {}^L\mathrm{Res}_{K/F} \mathrm{GL}_k, \text{ and } \begin{bmatrix} I_k & & \\ & \mathrm{GL}_{n-2k} & \\ & & I_k \end{bmatrix} \rtimes \{1, \jmath\} \cong {}^L U_{n-2k}.$$

Write

$$\bar\rho_M = \begin{bmatrix} \bar\rho_a & & \\ & \bar\rho_b & \\ & & \bar\rho_c \end{bmatrix} \rtimes *$$

By induction on the semisimple rank of G_n, we assume $\bar{\rho}_b$ admits a lift $\rho_b : \mathrm{Gal}_F \to {}^L U_{n-2k}(\bar{\mathbb{Z}}_p)$. Write $w = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix}$ for a longest Weyl group element. Let $(\rho_a, \rho_c) : \mathrm{Gal}_F \to {}^L\mathrm{Res}_{F/K}\,\mathrm{GL}_k(\bar{\mathbb{Z}}_p)$ be a potentially crystalline lift of $(\bar{\rho}_a, \bar{\rho}_c)$. We have

$$\rho_c(-) = J_3 \rho_a(\widehat{j} - \widehat{j}^{-1})^{-t} J_3^{-1}.$$

In particular, if λ is a potentially crystalline character with trivial mod p reduction, then $(\lambda\rho_a, \lambda^{-1}\rho_c)$ is another potentially crystalline lift of $(\bar{\rho}_a, \bar{\rho}_c)$. By choosing $\lambda := \bar{\mathbb{Z}}_p(n)$ with trivial reduction for n sufficiently large, we may assume the Hodge-Tate weights of $\mathrm{Hom}(\rho_b, \rho_a)$, $\mathrm{Hom}(\rho_c, \rho_a)$ and $\mathrm{Hom}(\rho_c, \rho_b)$ are all positive integers ≥ 2; in particular $H^1(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{Q}}_p)) = H^1_{\mathrm{crys}}(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{Q}}_p))$.

By Theorem 1, we can modify ρ_b without changing its Hodge-Tate weights and reduction mod p such that $\bar{\rho}|_{\mathrm{Gal}_K}$ admits a partial lift which is a partial extension of $(\rho_a|_{\mathrm{Gal}_K}, \rho_b|_{\mathrm{Gal}_K}, \rho_c|_{\mathrm{Gal}_K})$. Since K is a quadratic extension of F, $K \neq \mathbb{Q}_p$, the non-triviality of cup products follows from Lemma 6.6 and Proposition 6.3. Now the theorem follows from Theorem 5.5 and the main theorem of [L23D].

For the moreover part, note that we can choose ρ_a and ρ_b and ρ_c such that they are crystalline after restricting to Gal_K; if G is unramified, it means ρ_a and ρ_b and ρ_c are already crystalline. $\square$

8. Example B: symplectic groups

Since $G = \mathrm{GSp}_{2n}$ is a split group, we have $K = F$. The Dynkin diagram for G is . Thus the maximal proper Levi subgroups of G are of the form

$$\mathrm{GL}_k \times \mathrm{GSp}_{2(n-k)}, \text{ or } \mathrm{GL}_n.$$

Set

$$M_k := \begin{cases} \mathrm{GL}_k \times \mathrm{GSp}_{2(n-k)} & k < n \\ \mathrm{GL}_n & k = n \end{cases}$$

and write P_k for the corresponding parabolic subgroup.

Set $\Omega_k = \begin{bmatrix} & I_{n-k} \\ -I_{n-k} & \end{bmatrix}$. We use the following presentation of GSp_{2n}:

$$\mathrm{GSp}_{2n} = \{X \in \mathrm{GL}_{2n} | X^t \begin{bmatrix} & & I_k \\ & \Omega_k & \\ -I_k & & \end{bmatrix} X = \lambda \begin{bmatrix} & & I_k \\ & \Omega_k & \\ -I_k & & \end{bmatrix}\}$$

We have

$$P_k = \mathrm{GSp}_{2n} \cap \begin{bmatrix} \mathrm{GL}_k & \mathrm{Mat}_{k\times(2n-2k)} & \mathrm{Mat}_{k\times k} \\ & \mathrm{GL}_{2n-2k} & \mathrm{Mat}_{(2n-2k)\times k} \\ & & \mathrm{GL}_k \end{bmatrix} =: \mathrm{GSp}_{2n} \cap Q_k$$

where Q_k is the corresponding parabolic of GL_{2n}.

8.1. Lemma If $\bar{\rho} : \mathrm{Gal}_F \to \mathrm{GSp}_{2n}(\bar{\mathbb{F}}_p)$ is not elliptic, then there exists a parabolic P_k through which $\bar{\rho}$ factors and $\bar{\rho}$ is a Heisenberg-type extension of some $\bar{\rho}_M : \mathrm{Gal}_F \to M_k(\bar{\mathbb{F}}_p)$.

Proof. The proof is similar to that of Lemma 7.2. A parabolic $\bar{\rho}$ factors through P_k for some k. Write

$$\bar{\rho} = \begin{bmatrix} \bar{r}_1 & * & * \\ & \bar{r}_2 & * \\ & & \bar{r}_3 \end{bmatrix}$$

If $\bar{r}_1$ or $\bar{r}_3$ is not irreducible, then $\bar{\rho}$ also factors through P_s for some s strictly less than k. So we can assume both $\bar{r}_1$ and $\bar{r}_3$ are irreducible. Finally, local Tate duality ensures $\bar{\rho}$ is a Heisenberg-type extension. □

Write U and V for the unipotent radical of Q_k and P_k, respectively. We have

$$\mathrm{gr}^0 \operatorname{Lie} U = \mathrm{Mat}_{k\times k}$$

and

$$\mathrm{gr}^1 \operatorname{Lie} U = \mathrm{Mat}_{k\times 2(n-k)} \times \mathrm{Mat}_{2(n-k)\times k}$$

Define an $\Delta := \{1, \jmath\}$-action on $\operatorname{Lie} U$ by

$$\jmath(x,y) := (y^t\Omega_k, \Omega_k x^t), \quad (x,y) \in \mathrm{Mat}_{k\times 2(n-k)} \times \mathrm{Mat}_{2(n-k)\times k} \tag{3}$$

$$\jmath z := z^t, \quad z \in \mathrm{Mat}_{k\times k}\,. \tag{4}$$

8.2. Lemma We have $\operatorname{Lie} V = (\operatorname{Lie} U)^{\Delta}$.

Proof. Clear. □

8.3. Lemma The Δ-action on $\operatorname{Lie} U$ induces a classical Δ-action on $H^i(\mathrm{Gal}_K, \mathrm{gr}^j \operatorname{Lie} U(A))$ for each i, j, and $A = \bar{\mathbb{F}}_p,\ \bar{\mathbb{Z}}_p$.

Proof. Clear by Equation (3). □

8.4. Corollary For Galois representation $\rho_M = \begin{bmatrix} \tau_a & & \\ & \tau_b & \\ & & \tau_c \end{bmatrix} : \mathrm{Gal}_K \to M_k(\bar{\mathbb{Z}}_p)$. The cup product

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \operatorname{Lie} U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times c}(\bar{\mathbb{F}}_p))^{\Delta}$$

is non-trivial.

Proof. By Corollary 6.7 and Lemma 8.3, the cup product is non-trivial unless $K = \mathbb{Q}_p$, $k = 1$, and either

$$\bar{\tau}_b = \bar{\tau}_a(-1)^{\oplus 2n-2}, \text{ and } \bar{\tau}_c = \bar{\tau}_a(-1)$$

or

$$\bar{\tau}_b = \bar{\tau}_a^{\oplus 2n-2}, \text{ and } \bar{\tau}_c = \bar{\tau}_a(-1).$$

The symplecticity of ρ_M implies

$$\bar{\tau}_a\bar{\tau}_c = \lambda$$

and

$$\bar{\tau}_b^t \Omega_k \bar{\tau}_b = \lambda\Omega_k$$

where λ is the similitude character. Since $\bar{\tau}_b = \bar{\tau}_a(m)I_{2n-2}$ $(m = 0,\ -1)$ is forced to be a scalar matrix, we have

$$\bar{\tau}_a(m)^2 = \lambda.$$

Thus

$$\bar{\tau}_a^2(2m) = \bar{\tau}_a\bar{\tau}_c = \bar{\tau}_a(-1)$$

which implies $\bar{\mathbb{F}}_p(1) = \bar{\mathbb{F}}_p$, which contradicts the fact that $K = \mathbb{Q}_p$. □

8.5. Theorem Theorem 3 holds for ${}^L G_n = \mathrm{GSp}_{2n}$ and Sp_{2n}.

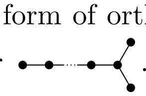
Proof. We will only treat GSp_{2n}; for Sp_{2n}, the reader can check that in the proof it is possible to ensure the similitude character is always 1. If $\bar{\rho}$ is elliptic, then it is [L23, Theorem C]. Suppose $\bar{\rho}$ is not elliptic, then by Lemma 8.1, there exists a parabolic P_k through which $\bar{\rho}$ factors through and that $\bar{\rho}$ is a Heisenberg-type extension of some $\bar{\rho}_M : \mathrm{Gal}_F \to M_k(\bar{\mathbb{F}}_p)$. Write

$$\bar{\rho}_M = \begin{bmatrix} \bar{\rho}_a & & \\ & \bar{\rho}_b & \\ & & \bar{\rho}_c \end{bmatrix}$$

By induction on the semisimple rank of G_n, we assume $\bar{\rho}_b$ admits a lift $\rho_b : \mathrm{Gal}_F \to \mathrm{GSp}_{n-2k}(\bar{\mathbb{Z}}_p)$ with similitude character μ. Let $\rho_a : \mathrm{Gal}_F \to \mathrm{GL}_k(\bar{\mathbb{Z}}_p)$ be a crystalline lift of $\bar{\rho}_a$. Set $\rho_c := \mu\rho_a^{-t}$. If λ is a potentially crystalline character with trivial mod p reduction, then $(\lambda\rho_a, \rho_b, \lambda^{-1}\rho_c)$ is a crystalline lift of $\bar{\rho}_M$. By choosing $\lambda = \bar{\mathbb{Z}}_p(n)$ with trivial reduction for n sufficiently large, we may assume the Hodge-Tate weights of $\mathrm{Hom}(\rho_b, \rho_a)$, $\mathrm{Hom}(\rho_c, \rho_a)$ and $\mathrm{Hom}(\rho_c, \rho_b)$ are all positive integers ≥ 2; in particular $H^1(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{Q}}_p)) = H^1_{\mathrm{crys}}(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{Q}}_p))$.

By Theorem 1, we can modify ρ_b without changing its Hodge-Tate weights and reduction mod p such that $\bar{\rho}$ admits a partial lift which is a partial extension of (ρ_a, ρ_b, ρ_c). By Corollary 8.4, the theorem follows from Theorem 5.5 and the main theorem of [L23D]. □

9. Example C: odd and even orthogonal groups

Let $G = \mathrm{GSO}_n$ be the split form of orthogonal similitude groups. We have $K = F$. The Dynkin diagram for G is or . Thus the maximal proper Levi subgroups of G are of the form

$$\mathrm{GL}_k \times \mathrm{GSO}_{n-2k},\ \text{ or }\ \mathrm{GL}_n\,.$$

Set

$$M_k := \begin{cases} \mathrm{GL}_k \times \mathrm{GSO}_{n-2k} & 2k < n \\ \mathrm{GL}_n & 2k = n \end{cases}$$

and write P_k for the corresponding parabolic subgroup.

We use the following presentation of GSO_n:

$$\mathrm{GSO}_n = \{X \in \mathrm{GL}_n \,|X^t \begin{bmatrix} & & I_k \\ & I_{n-2k} & \\ I_k & & \end{bmatrix} X = \lambda \begin{bmatrix} & & I_k \\ & I_{n-2k} & \\ I_k & & \end{bmatrix}\}$$

We have

$$P_k = \mathrm{GSO}_n \cap \begin{bmatrix} \mathrm{GL}_k & \mathrm{Mat}_{k\times(n-2k)} & \mathrm{Mat}_{k\times k} \\ & \mathrm{GL}_{n-2k} & \mathrm{Mat}_{(n-2k)\times k} \\ & & \mathrm{GL}_k \end{bmatrix} =: \mathrm{GSO}_n \cap Q_k$$

where Q_k is the corresponding parabolic of GL_n.

9.1. Lemma If $\bar{\rho} : \mathrm{Gal}_F \to \mathrm{GSO}_n(\bar{\mathbb{F}}_p)$ is not elliptic, then there exists a parabolic P_k through which $\bar{\rho}$ factors and $\bar{\rho}$ is a Heisenberg-type extension of some $\bar{\rho}_M : \mathrm{Gal}_F \to M_k(\bar{\mathbb{F}}_p)$.

Proof. It is completely similar to Lemma 8.1. □

Write U and V for the unipotent radical of Q_k and P_k, respectively. We have

$$\mathrm{gr}^0 \,\mathrm{Lie}\, U = \mathrm{Mat}_{k\times k}$$

and

$$\mathrm{gr}^1 \,\mathrm{Lie}\, U = \mathrm{Mat}_{k\times(n-2k)} \times \mathrm{Mat}_{(n-2k)\times k}$$

Define an $\Delta := \{1, \jmath\}$-action on $\mathrm{Lie}\, U$ by

$$\jmath(x,y) := \quad (-y^t, -x^t), \quad (x,y) \in \mathrm{Mat}_{k\times(n-2k)} \times \mathrm{Mat}_{(n-2k)\times k} \tag{5}$$

$$\jmath z := \quad -z^t, \quad z \in \mathrm{Mat}_{k\times k}\,. \tag{6}$$

9.2. Lemma We have $\mathrm{Lie}\, V = (\mathrm{Lie}\, U)^{\Delta}$.

Proof. Clear. □

9.3. Lemma The Δ-action on $\mathrm{Lie}\, U$ induces a classical Δ-action on $H^i(\mathrm{Gal}_K, \mathrm{gr}^j \,\mathrm{Lie}\, U(A))$ for each i, j, and $A = \bar{\mathbb{F}}_p,\ \bar{\mathbb{Z}}_p$.

Proof. Clear by Equation (5). □

9.4. Corollary For Galois representation $\rho_M = \begin{bmatrix} \tau_a & & \\ & \tau_b & \\ & & \tau_c \end{bmatrix} : \mathrm{Gal}_K \to M_k(\bar{\mathbb{Z}}_p)$. The cup product

$$H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \,\mathrm{Lie}\, U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \times H^1(\mathrm{Gal}_K, \mathrm{gr}^1 \,\mathrm{Lie}\, U(\bar{\mathbb{Z}}_p))^{\Delta} \underset{\bar{\mathbb{Z}}_p}{\otimes} \bar{\mathbb{F}}_p \to H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a\times c}(\bar{\mathbb{F}}_p))^{\Delta}$$

is non-trivial.

Proof. By Corollary 6.7 and Lemma 9.3, the cup product is non-trivial unless $K = \mathbb{Q}_p$, $k = 1$, and either

$$\bar{\tau}_b = \bar{\tau}_a(-1)^{\oplus 2n-2}, \text{ and } \bar{\tau}_c = \bar{\tau}_a(-1)$$

or

$$\bar{\tau}_b = \bar{\tau}_a^{\oplus 2n-2}, \text{ and } \bar{\tau}_c = \bar{\tau}_a(-1).$$

The orthogonality of ρ_M implies

$$\bar{\tau}_a \bar{\tau}_c = \lambda$$

and

$$\bar{\tau}_b^t \bar{\tau}_b = \lambda I_{n-2}$$

where λ is the similitude character. Since $\bar{\tau}_b = \bar{\tau}_a(m)I_{n-2}$ $(m = 0,\ -1)$ is forced to be a scalar matrix, we have

$$\bar{\tau}_a(m)^2 = \lambda.$$

Thus

$$\bar{\tau}_a^2(2m) = \bar{\tau}_a\bar{\tau}_c = \bar{\tau}_a(-1)$$

which implies $\bar{\mathbb{F}}_p(1) = \bar{\mathbb{F}}_p$, which contradicts the fact that $K = \mathbb{Q}_p$. □

9.5. Theorem Theorem 3 holds for ${}^L G_n = \mathrm{GSO}_n$ and SO_n.

Proof. We will only treat GSO_n; the reader can verify that in the proof it is possible to ensure the similitude character is always 1. If $\bar{\rho}$ is elliptic, then it is [L23, Theorem C]. Suppose $\bar{\rho}$ is not elliptic, then by Lemma 9.1, there exists a parabolic P_k through which $\bar{\rho}$ factors through and that $\bar{\rho}$ is a Heisenberg-type extension of some $\bar{\rho}_M : \mathrm{Gal}_F \to M_k(\bar{\mathbb{F}}_p)$. Write

$$\bar{\rho}_M = \begin{bmatrix} \bar{\rho}_a & & \\ & \bar{\rho}_b & \\ & & \bar{\rho}_c \end{bmatrix}$$

By induction on the semisimple rank of G_n, we assume $\bar{\rho}_b$ admits a lift $\rho_b : \mathrm{Gal}_F \to \mathrm{GSO}_{n-2k}(\bar{\mathbb{Z}}_p)$ with similitude character μ. Let $\rho_a : \mathrm{Gal}_F \to \mathrm{GL}_k(\bar{\mathbb{Z}}_p)$ be a crystalline lift of $\bar{\rho}_a$. Set $\rho_c := \mu\rho_a^{-t}$. If λ is a potentially crystalline character with trivial mod p reduction, then $(\lambda\rho_a, \rho_b, \lambda^{-1}\rho_c)$ is a crystalline lift of $\bar{\rho}_M$. By choosing $\lambda = \bar{\mathbb{Z}}_p(n)$ with trivial reduction for n sufficiently large, we may assume the Hodge-Tate weights of $\mathrm{Hom}(\rho_b, \rho_a)$, $\mathrm{Hom}(\rho_c, \rho_a)$ and $\mathrm{Hom}(\rho_c, \rho_b)$ are all positive integers ≥ 2; in particular $H^1(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{Q}}_p)) = H^1_{\mathrm{crys}}(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{Q}}_p))$.

By Theorem 1, we can modify ρ_b without changing its Hodge-Tate weights and reduction mod p such that $\bar{\rho}$ admits a partial lift which is a partial extension of (ρ_a, ρ_b, ρ_c). By Corollary 8.4, the theorem follows from Theorem 5.5 and the main theorem of [L23D]. □

10. The Emerton-Gee stacks for unitary groups

When we speak of

> "the locus of ... in the moduli stack of something",

we mean

> "the scheme-theoretic closure of the scheme-theoretic image of all families of something whose $\bar{\mathbb{F}}_p$-points are of the form ... in the moduli stack of something".

So a "locus" is techinically always a closed substack. However, since we are interested in dimension analysis only, it is almost always harmless to replace a "locus" by its dense open substacks.

When we speak of "the moduli of L-parameters", we always mean "the moduli of (φ, Γ)-modules" in the sense of [L23B].

When we write $H^\bullet(\mathrm{Gal}_F, -)$, we always mean (φ, Γ)-cohomology (or the cohomology of the corresponding Herr complex).

10.1. Theorem Let $\bar{\alpha} : \mathrm{Gal}_K \to \mathrm{GL}_a(\bar{\mathbb{F}}_p)$ be an irreducible Galois representation.

The locus of $\bar{x} \in \mathcal{X}_{F, {}^L U_n, \mathrm{red}}$ such that $\mathrm{Hom}_{\mathrm{Gal}_K}(\bar{\alpha}, \bar{x}|_{\mathrm{Gal}_K}) \geq r$ is of dimension at most

$$d_{n,r} := [F : \mathbb{Q}_p]\frac{n(n-1)}{2} - r^2 + \frac{r}{2}.$$

The whole section is devoted to the proof of Theorem 10.1. We denote by $\mathcal{X}_n^{\bar{\alpha}^{\oplus r}}$ the locus considered in Theorem 10.1. We will prove Theorem 10.1 by induction on n, and assume it holds for $n' < n$ throughout the section.

10.2. Involution of Galois representations Write $\Delta = \mathrm{Gal}(K/F) = \{1, \jmath\}$. For each irreducible representation $\beta : \mathrm{Gal}_K \to \mathrm{GL}_a(\bar{\mathbb{F}}_p)$, set $\theta(\beta) := \beta(\jmath^{-1} \circ - \circ \jmath)^{-t}$. Suppose

$$\begin{bmatrix} \bar{\alpha} & & \\ & \bar{\tau} & \\ & & \bar{\beta} \end{bmatrix} \rtimes *$$

is an L-parameter $\mathrm{Gal}_F \to {}^L M_a$. If $\jmath$ acts on GL_n via $x \mapsto w x^{-t} w^{-1}$, then direct computation shows

$$\bar{\beta} = b w \theta(\bar{\alpha}) w^{-1} b^{-1}$$

where w is the longest Weyl group element and $b = \bar{\beta}(\jmath)\jmath^{-1} \in \mathrm{GL}_a(\bar{\mathbb{F}}_p)$. In particular, $\bar{\beta}$ is isomorphic to $\theta(\bar{\alpha})$ as a Gal_K-representation.

10.3. Base case of the induction We first consider the elliptic locus of $\mathcal{X}_n^{\bar{\alpha}^{\oplus r}}$ (that is, the locus consisting of elliptic L-parameters). We only need to understand the case where $n = ra$.

Let $x : \mathrm{Gal}_F \to {}^L U_n(\bar{\mathbb{F}}_p)$ be an L-parameter in the elliptic locus of $\mathcal{X}_{ra}^{\bar{\alpha}^{\oplus r}}$. Since $x|_{\mathrm{Gal}_K} = \begin{bmatrix} \bar{\alpha}(-1) & & \\ & \dots & \\ & & \bar{\alpha}(-1) \end{bmatrix}$, it is immediate that we must have $\bar{\alpha}(-1) \cong \theta(\bar{\alpha}(-1))$. Moreover, $x(\jmath)$ permutes all diagonal blocks of $x|_{\mathrm{Gal}_K}$ without orbits of size 2; however, since all orbits have size at most 2, $x(\jmath)w^{-1}\jmath^{-1}$ is a diagonal matrix. We have the following:

10.4. Lemma If $\bar{\alpha}(-1) \ncong \theta(\bar{\alpha}(-1))$, the elliptic locus of $\mathcal{X}_{ra}^{\bar{\alpha}^{\oplus r}}$ is empty.

If $\bar{\alpha}(-1) \cong \theta(\bar{\alpha}(-1))$, the elliptic locus of $\mathcal{X}_{ra}^{\bar{\alpha}^{\oplus r}}$ has dimension at most $ra - r^2$. Moreover, if $r = 1$, then the elliptic locus has dimension -1.

Proof. We can take $x(\jmath)w^{-1}\jmath^{-1}$ to be any diagonal matrix (which completely determines x), and we can quotient out the block diagonal with scalar block entries. So there exists a surjective map from $[\mathrm{G}_m^{\times n} / \mathrm{GL}_r]$ to the elliptic locus of $\mathcal{X}_{ra}^{\bar{\alpha}^{\oplus r}}$ (here $\mathrm{G}_m^{\times n}$ parameterizes all diagonal matrices and GL_r parameterizes block matrices with $r \times r$ scalar matrix blocks). For the moreover part, note that if $r = 1$, then by Schur's lemma, $x(\jmath)w^{-1}\jmath^{-1}$ is a scalar matrix. Since $x(\jmath)^2 = x(\jmath^2) = \bar{\alpha}(-1)(\jmath^2)$ is fixed, $x(\jmath)$ has at most 2 choices while the automorphism of x include all scalar matrices and is at least one-dimensional.

Finally, we want to show $d_{ra,r} \geq ra - r^2$ of $r > 1$. We can rewrite it as $\frac{a[F:\mathbb{Q}_p]}{2}(ra-1) + 1/2 \geq a$. If $a > 1$, then $\frac{a[F:\mathbb{Q}_p]}{2}(ra-1) \geq \frac{3}{2}a > a$. If $a = 1$, then $\frac{a[F:\mathbb{Q}_p]}{2}(ra-1) + 1/2 \geq 1/2 + 1/2 = 1$. $\square$

10.5. The shape of $x \in \mathcal{X}_n^{\bar\alpha^{\oplus r}}$

If $\bar\alpha \not\cong \theta(\bar\alpha)$, then such x is of the form

$$(\dagger)\qquad \begin{bmatrix} \bar\alpha(-1)^{\oplus r} & * & * \\ & \bar\tau & * \\ & & \theta(\bar\theta(-1))^{\oplus r} \end{bmatrix} \rtimes *$$

where $\bar\tau \rtimes *$ is an L-parameter for U_{n-2ar}. If $\bar\alpha \cong \theta(\bar\alpha)$, then x can also be of the form

$$(\ddagger)\qquad \begin{bmatrix} \bar\alpha(-1)^{\oplus k_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \bar\alpha(-1)^{\oplus k_2} & * & 0 & * & * & 0 \\ & & \bar\tau_1 & 0 & * & * & 0 \\ & & & \bar\tau & 0 & 0 & 0 \\ & & & & \bar\tau_2 & * & 0 \\ & & & & & \bar\alpha(-1)^{\oplus k_2} & 0 \\ & & & & & & \bar\alpha(-1)^{\oplus k_1} \end{bmatrix} \rtimes *$$

where $\bar\tau$ is an elliptic L-parameter for U_{ak_3} and that $2k_1 + k_2 + k_3 = r$. Intuitively, the dimension of the locus of ($\dagger$) should be larger than the dimension of the locus of ($\ddagger$) because there are more zero entries in ($\ddagger$); the next lemma partially confirms our intuition and allows us to consider only ($\dagger$).

10.6. Lemma (1) Either $\dim \mathcal{X}_n^{\bar\alpha^{\oplus r}} \le d_{n,r}$, or the dimension of $\mathcal{X}_n^{\bar\alpha^{\oplus r}}$ is equal to the dimension of the locus of

$$\begin{bmatrix} \bar\alpha(-1)^{\oplus r} & * & * \\ & \bar\tau & * \\ & & \theta(\bar\alpha(-1))^{\oplus r} \end{bmatrix} \rtimes *,$$

where $\bar\tau \rtimes *$ is an L-parameter for U_{n-2ar}.

(2) The semisimple locus of $\mathcal{X}_n^{\bar\alpha^{\oplus r}}$ has dimension at most $d_{n,r}$.

Proof. (1) It is clear if $\theta(\bar\alpha(-1)) \not\cong \bar\alpha(-1)$. So, suppose $\theta(\bar\alpha(-1)) \cong \bar\alpha(-1)$.

Consider ($\ddagger$). There are two cases: $k_3 = 0$ and $k_3 \neq 0$.

We first assume $k_3 = 0$. Set $k = k_1$ and thus $k_2 = n - 2k$. Since

$$\mathrm{Aut}_{\mathrm{Gal}_K}(\bar\alpha^{\oplus k}) = \mathrm{Aut}_{\mathrm{Gal}_F}(\begin{bmatrix} \bar\alpha^{\oplus k} & \\ & \theta(\bar\alpha)^{\oplus k} \end{bmatrix} \rtimes *)$$

has dimension k^2, it remains to show

$$(7)\qquad -k^2 + d_{n-2ka, r-2k} \le d_{n,r},$$

which is equivalent to

$$(8)\qquad [F:\mathbb{Q}_p]a(-2ak + 2n - 1) - (4r - 5k - 1) \ge 0.$$

If $a = 1$, then the derivative of the LHS of (8) with respect to k is positive. So we only need to consider the $k = k_{\min} = 0$ case, which is clear. If $a > 1$, then the derivative with respect to k is negative, and we only need to consider the $k = k_{\max} = r$-case:

$$[F:\mathbb{Q}_p]a(-2ar + rn - 1) \ge -r - 1$$

whose LHS > 0 and RHS < 0.

Next assume $k_3 \neq 0$. Then

$$\begin{bmatrix} \bar{\alpha}(-1)^{\oplus k_2} & * & * & * \\ & \bar{\tau}_1 & * & * \\ & & \bar{\tau}_2 & * \\ & & & \bar{\alpha}(-1)^{\oplus k_2} \end{bmatrix} \rtimes *$$

is an L-parameter for $U_{n-2k_1a-k_3a}$. By reusing the inequality (7), it suffices to show

$$d_{n-2k_1a-k_3a,k_2} + d_{k_3a,k_3} \leq d_{n-2k_1a,r-2k_1} \tag{9}$$

We can set

$$n' = n - 2k_1a$$
$$r' = r - 2k_1$$
$$k' = k_2$$

and rename n', r', k' to n, r, k. By Lemma 10.4, (9) becomes

$$d_{n-(r-k)a,k} + (r-k)a - (r-k)^2 \leq d_{n,r} \tag{10}$$

Set $g(n,r,k) = d_{n,r} - (d_{n-(r-k)a,k} + (r-k)a - (r-k)^2)$. We have $\frac{\partial^2 g}{\delta r^2} = -[F:\mathbb{Q}_p]a^2/2 < 0$. Thus g achieves minimum at the boundary points of the range for r. Since $k \leq r \leq \frac{n}{a} - k$ and $g|_{r=k} = 0$, it suffices to show $f(n,k) = g(n, n/a - k, r) \geq 0$. We have $\frac{\partial^2 f}{\delta k^2} = 2(2 - [F:\mathbb{Q}_p]a^2)$. When $2 \leq [F:\mathbb{Q}_p]a^2$, f achieves minimum at the boundary points of the range for k. Since $0 \leq k \leq n/a$, $f|_{k=0} = \frac{n}{2a} + \frac{1}{2}n([F:\mathbb{Q}_p]n - [F:\mathbb{Q}_p] - 2) > 0$ (because $n \geq 3$), and $f|_{k=n/a} = 0$, (10) holds. When $2 > [F:\mathbb{Q}_p]a^2$, we must have $a = 1$ and $F = \mathbb{Q}_p$; now since $\frac{\partial f}{\partial k} = 4k - 2n + 2 < 0$, $f_{\min} = f|_{k=n/a} = 0$. So we are done.

(2) It has been implicitly proved in the proof of part (1). □

To analyze the locus of

$$\begin{bmatrix} \bar{\alpha}(-1)^{\oplus r} & * & * \\ & \bar{\tau} & * \\ & & \theta(\bar{\alpha}(-1))^{\oplus r} \end{bmatrix} \rtimes *,$$

we need to consider parabolic Emerton-Gee stacks.

10.7. Parabolic Emerton-Gee stacks Let A be a reduced finite type $\bar{\mathbb{F}}_p$-algebra. For any morphism $\operatorname{Spec} A \to \mathcal{X}_{{}^LM_{ra},\mathrm{red}}$, there is always a scheme-theoretically surjective morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that $\operatorname{Spec} B \to \mathcal{X}_{{}^LM_{ra},\mathrm{red}}$ is a basic morphism (see [L23B, Lemma 10.1.1, Definition 10.1.2]). Here B is also a reduced finite type $\bar{\mathbb{F}}_p$-algebra.

By replacing A by B, we assume $\operatorname{Spec} A \to \mathcal{X}_{{}^LM_{ra},\mathrm{red}}$ is a basic morphism. Then

$$\operatorname{Spec} A \times_{\mathcal{X}_{{}^LM_{ra},\mathrm{red}}} \mathcal{X}_{{}^LP_{ra}}$$

is an algebraic stack ([L23B, Proposition 10.1.8]).

Write $[U,U]$ for the derived subgroup of U, where U is the unipotent radical of ${}^LP_{ra}$. Note that $[U,U] \cong \mathrm{Mat}_{ra\times ra}$, $U^{\mathrm{ab}} := U/[U,U] \cong \mathrm{Mat}_{ra\times(n-2ra)} \oplus \mathrm{Mat}_{(n-2ra)\times ra}$.

For ease of notation, set

$$X_A := \operatorname{Spec} A,$$
$$Y_A := \operatorname{Spec} A \times_{\mathcal{X}_{{}^LM_{ra},\mathrm{red}}} \mathcal{X}_{{}^LP_{ra/[U,U]}},$$

and

$$Z_A := \operatorname{Spec} A \times_{\mathcal{X}_{{}^L M_{ra},\mathrm{red}}} \mathcal{X}_{{}^L P_{ra}}.$$

We can regard Y_A as a sheaf of groupoids over X_A. Denote by Y_A^C the coarse moduli sheaf (of sets) of Y_A over X_A. Then Y_A^C is representable by a scheme, and is a vector bundle over X_A of rank $H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(A))$, and

$$Y_A = [Y_A^C / H^0(\mathrm{Gal}_F, U^{\mathrm{ab}}(A))]$$

(see [L23B, Corollary 10.1.7]).

In like manner, denote by Z_A^C the coarse moduli sheaf of Z_A over X_A. Then Z_A^C is an affine bundle over Y_A^C of rank $H^1(\mathrm{Gal}_F, [U,U](A))$; and

$$Z_A = [Z_A^C / H^0(\mathrm{Gal}_F, [U,U](A)) \rtimes H^0(\mathrm{Gal}_F, U^{\mathrm{ab}}(A))].$$

Write W_A for the scheme-theoretic image of Z_A^C in Y_A^C. Indeed, the image of Z_A^C in Y_A^C is already closed and (the underlying set of) W_A is the set-theoretic image. Assume the codimension of W_A in Y_A^C is c. We have

$$\dim Z_A^C = \dim X_A - c + \operatorname{rank} H^1(\mathrm{Gal}_F, \operatorname{Lie} U(A)) \tag{11}$$

by the discussion above.

10.8. Lemma Let $\operatorname{Spec} A$ be an irreducible, finite type $\bar{\mathbb{F}}_p$-variety. Let $\operatorname{Spec} A \to \mathcal{X}_{F,{}^L M_r,\mathrm{red}}$ be a basic morphism of finite type such that each $x \in \operatorname{Spec} A(\bar{\mathbb{F}}_p)$ corresponds to an L-parameter of the form

$$\begin{bmatrix} \bar{\alpha}(-1)^{\oplus r} & & \\ & \bar{\tau} & \\ & & \theta(\bar{\alpha}(-1))^{\oplus r} \end{bmatrix} \rtimes *$$

such that $\bar{\tau} \in \mathcal{X}_{n-2r}^{\bar{\alpha}(-1)^{\oplus s}} \setminus \mathcal{X}_{n-2r}^{\bar{\alpha}(-1)^{\oplus(s+1)}}$.

Assume the scheme-theoretic image of $f : X_A \to \mathcal{X}_{F,{}^L M_{ra},\mathrm{red}}$ has dimension d_X, and the scheme-theoretic image of $g : Z_A \to \mathcal{X}_{F,{}^L U_n,\mathrm{red}}$ has dimension d_Z. Then

$$d_Z \leq d_X + rs - c + [F:\mathbb{Q}_p](2nra - 3r^2a^2) + \begin{cases} r^2 & \theta(\bar{\alpha}(-1)) = \bar{\alpha}(-2) \\ 0 & \theta(\bar{\alpha}(-1)) \neq \bar{\alpha}(-2). \end{cases}$$

We also have $d_X \leq d_{n-2ra,s} - r^2$.

Proof. Note that

- $\operatorname{rank} H^2(\mathrm{Gal}_F, [U,U](A)) \leq \begin{cases} r^2 & \theta(\bar{\alpha}(-1)) = \bar{\alpha}(-2) \\ 0 & \theta(\bar{\alpha}(-1)) \neq \bar{\alpha}(-2). \end{cases}$,
- $\operatorname{rank} H^2(\mathrm{Gal}_F, U^{\mathrm{ab}}(A)) \leq rs$ (see the sublemma below), and
- $\dim U = 2ra(n-2ra) + r^2a^2 = 2nra - 3r^2a^2$.

Sublemma $\operatorname{rank} H^2(\mathrm{Gal}_F, U^{\mathrm{ab}}(A)) \leq rs$.

Proof. It is clear that $\operatorname{rank} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{ra\times(n-2ra)}(A)) = \operatorname{rank} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{(n-2ra)\times ra}(A)) \leq rs$. Note that

$$H^2(\mathrm{Gal}_K, U^{\mathrm{ab}}(A)) = H^2(\mathrm{Gal}_K, \mathrm{Mat}_{ra\times(n-2ra)}(A)) \oplus H^2(\mathrm{Gal}_K, \mathrm{Mat}_{(n-2ra)\times ra}(A)),$$

and the $\mathrm{Gal}(K/F)$-action swaps the two direct summands (Lemma 7.3).

In particular, $\operatorname{rank} H^2(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^{\mathrm{Gal}(K/F)} \leq rs$. □

By the local Euler characteristic

$$\operatorname{rank} H^0(\mathrm{Gal}_F, \operatorname{Lie} U(A)) - \operatorname{rank} H^1(\mathrm{Gal}_F, \operatorname{Lie} U(A)) + \operatorname{rank} H^2(\mathrm{Gal}_F, \operatorname{Lie} U(A)) = -[F:\mathbb{Q}_p] \dim U$$

we have

$$\begin{aligned}\operatorname{rank} H^1(\mathrm{Gal}_F, \operatorname{Lie} U(A)) \leq [F:\mathbb{Q}_p](2nra - 3r^2a^2) + \begin{cases} r^2 & \theta(\bar\alpha(-1)) = \bar\alpha(-2) \\ 0 & \theta(\bar\alpha(-1)) \neq \bar\alpha(-2)\end{cases} \\ + rs + \operatorname{rank} H^0(\mathrm{Gal}_F, \operatorname{Lie} U(A)).\end{aligned}$$

By Equation (11), it suffices to show

$$d_Z - d_X \leq \dim Z_A^C - \dim X_A - \operatorname{rank} H^0(\mathrm{Gal}_F, \operatorname{Lie} U(A)). \tag{12}$$

Let $W_A' \subset W_A$ be an irreducible component of largest dimension (see [Stacks, 0DR4] if the reader is not familiar with irreducible components of algebraic stacks). Set $(Z_A^C)' := Z_A^C \times_{W_A} W_A'$. We have $\dim Z_A^C = \dim (Z_A^C)'$ since $Z_A^C \to W_A$ is an affine bundle. Moreover, since W_A has only finitely many irreducible components, we can assume d_Z is the dimension of the scheme-theoretic image of $(Z_A^C)'$ in $\mathcal{X}_{F, {}^L U_n, \mathrm{red}}$. So, after suitable replacements, it is harmless to assume W_A, X_A and Z_A^C are irreducible varieties when proving (12). Now we can invoke [Stacks, Lemma Tag 0DS4]: after replacing X_A by a dense open (which does not change any quantity in (12) by the irreducibility of X_A), we can assume $\dim_t (X_A)_{f(t)} = \dim X_A - d_X$ for all $t \in W_A(\bar{\mathbb{F}}_p)$; similarly, after replacing Z_A^C by a dense open, we can assume $\dim_x (Z_A^C)_{g(x)} = \dim Z_A^C - d_Z$ for all $x \in Z_A^C(\bar{\mathbb{F}}_p)$. Label the projection $Z_A^C \to W_A$ by π. Now (12) becomes

$$\dim_{\pi(x)} (X_A)_{f(\pi(x))} \leq \dim_x (Z_A^C)_x - \operatorname{rank} H^0(\mathrm{Gal}_F, \operatorname{Lie} U(A)).$$

Denote by $G_{\pi(x)} \subset (\widehat{M_{ra}})_{\bar{\mathbb{F}}_p}$ and $G_x \subset (\widehat{U_n})_{\bar{\mathbb{F}}_p}$ the automorphism group of the L-parameters corresponding to $\pi(x)$ and x, resp.. Note that the immersion $[\operatorname{Spec} \bar{\mathbb{F}}_p / G_{\pi(x)}] \hookrightarrow \mathcal{X}_{F, {}^L M_{ra}, \mathrm{red}}$ induces an immersion $[(X_A)_{f(\pi(x))}/G_{\pi(x)}] \hookrightarrow X_A$. Similarly, we have an immersion $[(Z_A^C)_x/G_x] \hookrightarrow Z_A^C$. Note that the image of the composite $[(Z_A^C)_x/G_x] \hookrightarrow Z_A^C \to X_A$ contains the image of $[(X_A)_{f(\pi(x))}/G_{\pi(x)}] \hookrightarrow X_A$ (by, for example, the moduli interpretation), and therefore $\dim[(Z_A^C)_x/G_x] \geq \dim[(X_A)_{f(\pi(x))}/G_{\pi(x)}]$. Since

$$\begin{aligned}\dim[(Z_A^C)_x/G_x] &= \dim (Z_A^C)_x - \dim G_x \\ \dim[(X_A)_{f(\pi(x))}/G_{\pi(x)}] &= \dim (X_A)_{f(\pi(x))} - \dim G_{\pi(x)},\end{aligned}$$

it remains to show

$$\dim G_{\pi(x)} \leq \dim G_x - \operatorname{rank} H^0(\mathrm{Gal}_F, \operatorname{Lie} U(A)),$$

which is clear since $G_{\pi(x)} \rtimes (H^0(\mathrm{Gal}_F, [U,U]) \rtimes H^0(\mathrm{Gal}_F, U^{\mathrm{ab}})) \subset G_x$.

Finally, $d_X \leq d_{n-2r,s} - r^2$ is clear since

$$\mathrm{Aut}_{\mathrm{Gal}_K}(\bar\alpha^{\oplus r}) = \mathrm{Aut}_{\mathrm{Gal}_F}(\begin{bmatrix} \bar\alpha^{\oplus r} & \\ & \theta(\bar\alpha)^{\oplus r}\end{bmatrix} \rtimes *)$$

has dimension r^2. □

10.9. Initial estimates To prove Theorem 10.1, we want to show $d_Z \leq d_{n,r}$ for all $0 \leq s \leq \frac{n-2r}{2a}$. By Lemma 10.8, it suffices to prove

$$(d_{n-2r,s} - r^2) + r^2 + rs - c + [F : \mathbb{Q}_p](2nra - 3r^2a^2) \leq d_{n,r}. \tag{13}$$

for all $0 \leq s \leq \frac{n-2r}{2a}$. Expanding (13), we get

$$r^2 + rs - s^2 + \frac{s-r}{2} \leq [F : \mathbb{Q}_p](r^2 - r) + c. \tag{14}$$

Regarding the LHS of (13) as a quadratic polynomial in s, it achieves maximum at $s = r/2 + 1/4$; since s only takes integral values, we only need to prove (13) for $s = r/2$:

$$\frac{5}{4}r^2 - \frac{1}{4}r \leq [F : \mathbb{Q}_p](a^2r^2 - ar) + c \tag{15}$$

Using the trivial estimate $c \geq 0$, we only need to prove

$$\frac{5}{4}r - \frac{1}{4} \leq [F : \mathbb{Q}_p](a^2r - a) \tag{16}$$

which clearly holds when $a \geq 2$.

10.10. Lemma (The codimension lemma) Write h for rank $H^1(\mathrm{Gal}_K, \bar{\tau} \otimes \theta(\bar{\alpha}(-1))^\vee)$. Assume

- $H^2(\mathrm{Gal}_K, [U, U](A)) \neq 0$ and
- $a = 1$.

Then

$$c \geq c(r, h) := \min_{1 \leq k \leq r} \frac{1}{2}(k^2 + hr - hk) = \begin{cases} r^2/2 & h > 2r \\ (hr - h^2/4)/2 & h \leq 2r. \end{cases}$$

Note that the minimum of $C = \min_{1 \leq k \leq r} \frac{1}{2}(k^2 + hr - hk)$ is achieved at either $k = r$ or $k = h/2$. When $k = r$, the $C = r^2/2$. When $k = h/2$, $C = \frac{1}{2}(hr - h^2/4) \leq r^2/2$.

We will postpone the proof of the codimension lemma to the end of this section. Next, we prove that the codimension lemma implies Theorem 10.1.

Proof of Theorem 10.1. We've already settled the $a > 1$ case. So, assume $a = 1$.

If $h > 2r$, after plugging the codimension lemma 10.10 into (15), we only need to prove

$$\frac{5}{4}r^2 - \lceil \frac{1}{2}r^2 \rceil - \frac{1}{4}r \leq [F : \mathbb{Q}_p](a^2r^2 - ar) \tag{17}$$

which holds for all integers r and a. We remark that the inequality (17) is tight, and it achieves equality when $r = 1$, $a = 1$, and $[F : \mathbb{Q}_p]$ arbitrary.

Suppose $h \leq 2r$. If $n \leq 3r$, then the LHS (14) achieves maximum at $s = \frac{n-2r}{2}$. So we need to prove

$$\lfloor r^2 + r\frac{n-2r}{2} - (\frac{n-2r}{2})^2 + \frac{n-2r-r}{2} - \frac{hr - h^2/4}{2} \rfloor \leq [F : \mathbb{Q}_p](r^2 - r) \tag{18}$$

because the dimension only takes integral values. By the local Euler characteristic, $2r \geq h \geq n-2r+1$; and thus the LHS of (18) achieves minimum at $h = n - 2r + 1$. So we get

$$\lfloor -n^2/8 + nr/2 + r^2/2 + n/2 - 2r + 1/8 \rfloor \leq [F : \mathbb{Q}_p](r^2 - r), \tag{19}$$

whose LHS achieves maximum at the critical point $n = 2r + 2$:

$$\lfloor r^2 - r + 5/8 \rfloor \leq [F : \mathbb{Q}_p](r^2 - r) \tag{20}$$

which clearly holds.

If $n > 3r$, then the LHS (14) achieves maximum at $s = r/2$. So we need to prove

$$\lfloor \frac{5}{4}r^2 - \frac{1}{4}r - \frac{hr - h^2/4}{2} \rfloor \leq [F : \mathbb{Q}_p](r^r - r). \tag{21}$$

By the local Euler characteristic, $2r \geq h \geq n - 2r + 1$; and thus the LHS of (18) achieves minimum at $h = n - 2r + 1$. So we get

$$\lfloor n^2/8 - nr + 11r^2/4 + n/4 - 5r/4 + 1/8 \rfloor \leq [F : \mathbb{Q}_p](r^2 - r), \tag{22}$$

whose LHS achieves maximum at the boundary point $n = 3r + 1$:

$$\lfloor \frac{7}{8}r^2 - \frac{3}{4}r + 1/2 \rfloor \leq [F : \mathbb{Q}_p](r^2 - r), \tag{23}$$

which clearly holds. □

Finally, we turn to the codimension lemma.

10.11. Lemma Let κ be a field. Let $f_1(\mathbf{x}), \dots, f_m(\mathbf{x}) \in \kappa[\mathbf{x}] := \kappa[x_1, \dots, x_n]$ be homogeneous polynomials in n variables. Define

$$X = \operatorname{Spec} \kappa[\mathbf{x}]/(f_1(\mathbf{x}), \dots f_m(\mathbf{x}))$$

and

$$Y = \operatorname{Spec} \kappa[\mathbf{x}, \mathbf{y}]/(f_1(\mathbf{x}) - f_1(\mathbf{y}), \dots, f_m(\mathbf{x}) - f_m(\mathbf{y})).$$

Then $\dim X \leq \frac{1}{2} \dim Y$. Equivalently, the codimension of X in $\operatorname{Spec} \kappa[\mathbf{x}]$ is at least half of the codimension of Y in $\operatorname{Spec} \kappa[\mathbf{x}, \mathbf{y}]$.

Proof. There exists an embedding $X \times X \hookrightarrow Y$, $(\mathbf{x}, \mathbf{x}') \mapsto (\mathbf{x}, \mathbf{y})$. □

We remark that the inequality in Lemma 10.11 is sharp. If $X = \operatorname{Spec} \mathbb{F}[x_1, x_2, x_3]/(x_1x_2, x_1x_3)$ and $Y = \operatorname{Spec} \mathbb{F}[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1x_2 - y_1y_2, x_1x_3 - y_1y_3)$, then $\dim X = 2$ and $\dim Y = 4$.

10.12. Lemma Let $Y \to X$ be a morphism of finite type schemes over a field. Then there exists a point $x : \operatorname{Spec} \kappa \to X$ such that $\dim Y - \dim X \leq \dim Y \times_{X,x} \operatorname{Spec} \kappa$.

In particular, if $Y' \subset Y$ is a closed subscheme, then the codimension of Y' in Y is at least as large as the largest codimension of $Y' \times_{X,x} \kappa$ in $Y \times_{X,x} \operatorname{Spec} \kappa$ for some point x.

Proof. It is a standard fact. See, for example, [Stacks, Tag 0DS4]. □

Recall that

$$H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A)) = H^1(\mathrm{Gal}_K, \mathrm{Mat}_{ra \times (n-2ra)}(A)) \oplus H^1(\mathrm{Gal}_K, \mathrm{Mat}_{(n-2ra) \times ra}(A)),$$

and the $\mathrm{Gal}(K/F) = \{1, \jmath\}$-action swaps the two direct summands. We can also decompose $H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))$ according to the eigenvalues of $\jmath$:

$$H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A)) = H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^+ \oplus H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^-$$

where

$$H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^+ = \{x^+ := (x, \jmath x)\} = H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(A))$$
$$H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^- = \{x^- := (x, -\jmath x)\}$$

for $x \in H^1(\mathrm{Gal}_K, \mathrm{Mat}_{ra\times(n-2ra)}(A))$. There exists a bijection

$$H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^+ \xrightarrow{x^+\mapsto x^-} H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^-.$$

Note that

$$x^+ \cup x^+ = 2(x,0) \cup (0, \jmath x)$$
$$x^- \cup x^- = -2(x,0) \cup (0, \jmath x)$$

and thus $x^+ \cup x^+ = -x^- \cup x^-$. The upshot is there exists an isomorphism respecting cup products

$$H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A)) \to H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(A)) \oplus H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(A));$$

here we define cup products on the RHS by $(a,b)\cup(c,d) = a\cup c - b \cup d$.

Proof of the codimension lemma 10.10. By Lemma 10.11, Lemma 10.12 and the discussion above, it suffices to show for each $\bar{\mathbb{F}}_p$-point of A, the codimension of the locus $W^C := \{x \in H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p)) | x \cup x = 0\}$ in $H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p))$ (when regarded as a vector bundle over $\mathrm{Spec}\,\bar{\mathbb{F}}_p$) is at least $2c(r,h)$; as it forces the codimension of $\{x \in H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p)) | x\cup x = 0\}$ in $H^1(\mathrm{Gal}_F, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p))$ to be at least $c(r,h)$.

Consider all extensions of Gal_K-modules

$$\begin{bmatrix} \bar\alpha(-1)^{\oplus r} & * & * & * & * \\ & \bar\alpha(-2)^{\oplus s} & ? & ? & * \\ & & ? & ? & * \\ & & & \bar\alpha(-1)^{\oplus s} & * \\ & & & & \bar\alpha(-2)^{\oplus r} \end{bmatrix} =: \begin{bmatrix} \bar\alpha(-1)^{\oplus r} & * & * \\ & \bar\tau & * \\ & & \bar\alpha(-2)^{\oplus r} \end{bmatrix} =: \begin{bmatrix} \bar\alpha(-1)^{\oplus r} & * \\ & \bar\eta \end{bmatrix} =: \bar\rho$$

where ? means fixed and $*$ means undetermined. The coarse moduli space Y^C of all extensions modulo $[U,U]$ is the vector space $H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\bar{\mathbb{F}}_p))$.

There is another way to think about extensions. We can first extend $\bar\alpha(-1)^{\oplus r} \oplus \bar\tau \oplus \bar\alpha(-2)^{\oplus r}$ to $\bar\alpha(-1)^{\oplus r} \oplus \bar\eta$, and then extend $\bar\alpha(-1)^{\oplus r} \oplus \bar\eta$ to $\bar\rho$.

The coarse moduli space T^C of all extensions $\bar\eta$ is the vector space $H^1(\mathrm{Gal}_K, \bar\tau \otimes \bar\alpha(-2)^{\oplus r\vee})$. Denote by $T^C_k \subset T^C$ the subvariety consisting of $\bar\eta$ such that

$$\dim H^2(\mathrm{Gal}_K, \bar\alpha(-1)\otimes \bar\eta^\vee) - \dim H^2(\mathrm{Gal}_K, \bar\alpha(-1)\otimes\bar\tau^\vee) = r-k.$$

Write Z^C for the coarse moduli space of all extensions $\bar\rho$. Set $Z^C_k := Z^C \times_{T^C} T^C_k$. If $\bar\eta = \begin{bmatrix} \bar\tau & * \\ & \bar\alpha(-2)^{\oplus r} \end{bmatrix}$ lies in T^C_k, then the column space of $*$ is k-dimensional. To specify a point of T^C_k is the same as specifying a point of the Grassmannian $\mathrm{Gr}(k,r)$ and a point of $H^1(\mathrm{Gal}_K, \bar\tau\otimes\bar\alpha(-2)^{\oplus k\vee})$:

$$\begin{aligned} \dim T^C_k &\le \dim \mathrm{Gr}(k,r) + kh \\ &= k(r-k) + kh \\ &= \dim H^1(\mathrm{Gal}_K, \bar\tau\otimes\bar\alpha(-2)^{\oplus r\vee}) - rh + k(r-k) + kh. \end{aligned}$$

Note that there exists a stratification of locally closed subvarieties of T^C_k such that $H^\bullet(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes \bar\eta^\vee)$ has constant dimension over each stratum. After replacing T^C_k by the disjoint union of its stratums,

Z_k^C is a vector bundle over T_k^C of rank

$$\begin{aligned}\dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\eta^\vee) &= [K:\mathbb{Q}_p]r(n-2r) + \dim H^0(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\eta^\vee)\\ &\quad + \dim H^2(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\eta^\vee)\\ &\le \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\tau^\vee) + \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\alpha(-2)^{\oplus r\vee})\\ &\quad + \dim H^2(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\eta^\vee) - \dim H^2(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\tau^\vee)\\ &\quad - \dim H^2(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\alpha(-2)^{\oplus r\vee})\\ &= \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\tau^\vee) + \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\alpha(-2)^{\oplus r\vee})\\ &\quad + r(r-k) - r^2\\ &= \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\tau^\vee) + \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\alpha(-2)^{\oplus r\vee}) - kr.\end{aligned}$$

Therefore

$$\begin{aligned}\dim Z_k^C &= \dim T_k^C + \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\eta^\vee)\\ &\le \dim H^1(\mathrm{Gal}_K, \bar\tau\otimes\bar\alpha(-2)^{\oplus r\vee}) - rh + k(r-k) + kh\\ &\qquad \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\tau^\vee) + \dim H^1(\mathrm{Gal}_K, \bar\alpha(-1)^{\oplus r}\otimes\bar\alpha(-2)^{\oplus r\vee}) - kr\\ &= \dim H^1(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{F}}_p)) - rh + k(r-k) + kh - kr\\ &= \dim H^1(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{F}}_p)) - rh - k^2 + kh.\end{aligned}$$

Since Z^C is the union of Z_k^C, we have

$$\begin{aligned}\dim Z^C &\le \dim H^1(\mathrm{Gal}_K, \mathrm{Lie}\, U(\bar{\mathbb{F}}_p)) - \min_k(rh + k^2 - kh)\\ &= \dim Y^C + \dim H^1(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p)) - \min_k(rh + k^2 - kh).\end{aligned}$$

On the other hand, Z^C is an affine bundle over W^C of rank $\dim H^1(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p))$. Thus

$$\dim Z_C \ge \dim W_C + \dim H^1(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p)).$$

Finally,

$$\begin{aligned}\dim Y^C - \dim W_C &\ge \dim Y^C - (\dim Z_C - \dim H^1(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p)))\\ &\ge \dim Y^C - (\dim Y^C + \dim H^1(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p)) - \min_k(rh + k^2 - kh)\\ &\qquad - \dim H^1(\mathrm{Gal}_K, [U,U](\bar{\mathbb{F}}_p)))\\ &= \min_k(rh + k^2 - kh)\end{aligned}$$

□

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