A DELIGNE-LUSZTIG TYPE CORRESPONDENCE FOR TAME p-ADIC GROUPS

LIN, ZHONGYIPAN

Abstract. We establish a "matrix simultaneous diagonalization theorem" for disconnected reductive groups which relaxes both the semisimplicity condition and the commutativity condition. As an application, we prove the following basic results concerning mod p Langlands parameters for quasi-split tame groups G over a p-adic field F:

- All semisimple L-parameters $\mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$ factor through the L-group of a maximal F-torus of G;
- All semisimple mod p L-parameters admit a de Rham lift of regular p-adic Hodge type;
- A version of tame inertial local Langlands correspondnece; and
- A group-theoretic description of irreducible components of the reduced Emerton-Gee stacks away from Steinberg parts.

We also propose generalizations of the explicit recipe for Serre weights (after Herzig) and the geometric Breuil-Mézard for tame groups.

Contents

Date: October 10, 2023.

1. Introduction

Contents

In this paper, we present (without proving) novel extensions of both the explicit Serre weight conjecture and the Breuil-Mézard conjecture. Prior to stating these conjectures, we establish foundational results concerning mod p Langlands parameters for tame groups, such as the classification of elliptic Langlands parameters, the construction of their de Rham lifts, and a version of tame inertial local Langlands correspondence. The conjectures we formulate interpolate the respective conjectures for ramified general linear groups $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_n$ (as discussed in [EG23] and [LLHLM22]) and unramified reductive groups (as discussed in [GHS] and [FLH]).

To set the stage, let us denote by F a finite extension of $\mathbb{Q}_p$ with residue field κ_F. Write $\mathrm{Gal}_F := \mathrm{Gal}(F^s/F)$ for the absolute Galois group of F, and W_F for the Weil group of F. Let G be a quasi-split group over F which splits over a tame extension L of $\mathbb{Q}_p$ (so F is also a tame extension of $\mathbb{Q}_p$). Write ${}^L G = \widehat{G} \rtimes \mathrm{Gal}(L/F)$ for the L-group of G.

1.1. Motivation In [Se87], Serre formulated a precise conjecture predicting the minimal weight of mod p Galois representations arising from modular eigenforms. He also posed a question regarding the potential connection between the weight recipe and a “mod p Langlands philosophy,” as well as whether the weight recipe generalizes to encompass general reductive groups (Question 3.4, loc. cit.).

Subsequently, extensive research has been conducted on the Serre weight conjectures, with references available in the introduction of [GHS]. The modern version of Serre weight conjecture seeks to classify congruences of Hecke eigensystems in the cohomology of locally symmetric space associated to a reductive group G, with coefficients in local systems induced by different weights of G.

Since the formulation of the Serre weight conjecture for unramified groups in [GHS], substantial progress has been made in verifying cases of the conjecture beyond GL_n, particularly for groups such as GSp_4 ([Lee23]) and the unramified quasisplit unitary group U_2 ([KM22]). These accomplishments motivate further investigation into mod p Langlands parameters and their associated moduli stacks.

In this paper, as well as in the companion paper [L23C], we undertake a systematic classification of mod p Langlands parameters and explore their lifts in characteristic 0.

1.2. Mod p Langlands parameters An L-parameter $\bar\rho : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ is said to be *parabolic* if it factors through a parabolic subgroup ${}^L P(\bar{\mathbb{F}}_p)$, and is said to be *elliptic* if otherwise. Here ${}^L P := \widehat{P} \rtimes \mathrm{Gal}(L/F) \subset {}^L G$ and $\widehat{P}$ is a $\mathrm{Gal}(L/F)$-stable parabolic subgroup of $\widehat{G}$. An elliptic L-parameter is *semisimple* in the sense that its image is a completely reducible subgroup of ${}^L G(\bar{\mathbb{F}}_p)$.

The first main theorem we prove is the following characterization of elliptic mod p L-parameters.

Theorem A. *(Theorem 3.4.1) If $\bar{\rho} : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ is semisimple, then there exists a maximally unramified maximal F-torus S of G such that ρ admits a factorization*

$$\mathrm{Gal}_F \xrightarrow{\bar{\rho}_S} {}^L S(\bar{\mathbb{F}}_p) \xrightarrow{{}^L j} {}^L G(\bar{\mathbb{F}}_p)$$

where ${}^L j$ is the mod p Langlands-Shelstad L-embedding. Moreover, if ρ is elliptic, then S is an elliptic torus.

The immediate consequence of the mod p Langlands-Shelstad factorization theorem is the existence of de Rham lifts $\rho : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{Z}}_p)$. Indeed, to construct a de Rham lift of $\bar{\rho}$, it suffices to construct a lift of $\bar{\rho}_S$, which can be done via the p-adic Local Langlands Correspondence for algebraic tori (see [Ch20]). In Section 5.2, we discuss the p-adic Hodge-theoretic refinement of the LLC for algebraic tori and prove the following.

Theorem B. *(Theorem 5.3.1) If $\bar{\rho} : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ is semisimple, then there exists a potentially crystalline lift of $\bar{\rho}$ of regular Hodge type.*

If G is a ramified group, then there is no semistable or crystalline L-parameters; therefore potentially crystalline lifts are the best we can hope for. Theorem A plays a pivotal role in [L23], where it is used as a crucial input for establishing the Noetherian formal algebraicity of the Emerton-Gee stack $\mathcal{X}_{{}^L G}$ which is the foundation of many recent developments such as [LLHLM23] and [FLH].

We delve into the theory of parabolic/non-semisimple Langlands parameters in the companion paper [L23C]. For classical groups, a parabolic mod p L-parameter is an iterated Heisenberg-type extension of elliptic L-paramaters. Based on Theorem B, we reduce the general existence of de Rham lifts to a question about the dimension of certain closed substacks of the reduced Emerton-Gee stacks, and answer this question affirmatively for unitary groups using the geometry of Grassmannian manifolds in loc. cit.. So, for tamely ramified unitary groups, Theorem B holds for any L-parameter $\bar{\rho}$, not just semisimple ones (see Theorem 5, loc.cit.).

1.3. Parahoric Serre weights and the qualitative Breuil-Mézard conjecture In this subsection, we briefly discuss implications of the existence of de Rham lifts.

The Emerton-Gee stacks The Emerton-Gee stacks are a version of moduli stacks of L-parameters that enables us to apply the powerful machinery of geometric representation theory to the study of Galois deformation rings. They are first constructed for GL_n in [EG23], and then generalized to general tame groups in [L23].

The GL_n-case In [EG23], the irreducible components of the reduced Emerton-Gee stack $\mathcal{X}_{\mathrm{GL}_n,\mathrm{red}}$ are shown to be in bijection with isomorphism classes of irreducible $\bar{\mathbb{F}}_p$-representations of the finite group $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}(\mathbb{F}_p)$. An irreducible $\bar{\mathbb{F}}_p$-representation of the finite group $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}(\mathbb{F}_p)$ can be inflated to an irreducible $\bar{\mathbb{F}}_p$-representation of the compact group $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}(\mathbb{Z}_p)$, which is a superspecial parahoric subgroup of $\mathrm{GL}(F)$. The subgroup $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}(\mathbb{Z}_p)$ is also a maximally bounded subgroup of $\mathrm{GL}(F)$.

The tame group case In general, a parahoric subgroup of $G(F)$ is not necessarily a maximally bounded subgroup of $G(F)$. We fix a superspecial parahoric $\mathcal{G}^\circ$ of $G(F)$ and let $\mathcal{G} \supset \mathcal{G}^\circ$ be the maximally bounded subgroup containing $\mathcal{G}^\circ$. Write $\underline{G}$ for the reductive quotient of $\mathcal{G}^\circ$, which is a connected reductive group over the finite field κ_F. We call isomorphism classes of irreducible $\bar{\mathbb{F}}_p$-representations of $\mathcal{G}^\circ$ *parahoric Serre weights*. A *Serre weight* is defined to be an isomorphism class of pairs $(\sigma, \widetilde{\sigma})$ where $\widetilde{\sigma}$ is an irreducible $\bar{\mathbb{F}}_p$-representations of $\mathcal{G}$ and $\sigma \subset \widetilde{\sigma}$ is an irreducible $\bar{\mathbb{F}}_p$-representations of $\mathcal{G}^\circ$. By abuse of notation, we write $\widetilde{\sigma}$ for the pair $(\sigma, \widetilde{\sigma})$.

Conjecturally, for each Serre weight $\widetilde{\sigma}$, we can attach to it a finite union of irreducible components of $\mathcal{X}_{{}^LG,\mathrm{red}}$ which we denote by $\mathcal{C}(\widetilde{\sigma})$, such that

$$\mathcal{X}_{{}^LG,\mathrm{red}} = \cup_{\widetilde{\sigma}} \mathcal{C}(\widetilde{\sigma})$$

and $\mathcal{C}(\widetilde{\sigma}) \cap \mathcal{C}(\widetilde{\sigma}')$ is nowhere dense in $\mathcal{X}_{{}^LG,\mathrm{red}}$ if $\widetilde{\sigma} \not\cong \widetilde{\sigma}'$. Moreover, if $\underline{G}$ has simply-connected derived subgroup, we cojecture that each $\mathcal{C}(\widetilde{\sigma})$ is an irreducible component of $\mathcal{X}_{{}^LG,\mathrm{red}}$. This is the natural generalization of the so-called *qualitative Breuil-Mézard conjecture*, see [EG23, Section 8.1]. The content of the qualitative Breuil-Mézard conjecture is that $\mathcal{X}_{{}^LG,\mathrm{red}}$ is equidimensional of the same dimension as the special fiber potentially semistable/crystalline deformation rings and its irreducible components admit a group-theoretic parameterization. (In the literature, the qualitative Breuil-Mézard conjecture sometimes refers to a stronger statement, see [LLHLM23, Theorem 1.4.5], which further claims that a certain transition matrix is upper triangular. See Subsection 1.7 below for details.) If σ is a parahoric Serre weight, we can define a closed substack $\mathcal{C}(\sigma) = \cup_{(\sigma,\widetilde{\sigma})} \mathcal{C}(\widetilde{\sigma})$. We will postpone the construction of $\sigma \mapsto \mathcal{C}(\sigma)$ for *regular* parahoric Serre weights to Subsection 1.7 because we haven't introduced the necessary notations yet.

For classical groups, the qualitative Breuil-Mézard conjecture follows from the existence of de Rham lifts of regular Hodge type, see [L23B]. In loc. cit., we prove the qualitive Breuil-Mézard conjecture for unitary groups; in particular, for even unitary groups, the irreducible components of $\mathcal{X}_{{}^LU,\mathrm{red}}$ are in bijection with Serre weights (which, in this particular case, coincide with parahoric Serre weights). We also note that the association $\sigma \mapsto \mathcal{C}(\sigma)$ is constructed for all parahoric Serre weights in loc. cit., not just the regular ones.

1.4. Simultaneous diagonalization of matrices Before we proceed to formulate the conjectures, we digress and explain the proof of Theorem A.

Characteristic p versus characteristic 0 coefficients The characteristic 0 coefficient version of Theorem A essentially follows from the work of Borel-Serre on solvable subgroups of compact Lie groups ([BS53]). Some variants of the argument can be found in the literature; see, for example, [Kal19b]. However, these results make use of the assumption that the image of semisimple/elliptic $\bar{\rho}$ consists of semisimple elements, which is not true in characteristic p. For example, consider $\mathrm{Ind}_{\mathbb{Q}_4}^{\mathbb{Q}_2} 1 : \mathrm{Gal}_{\mathbb{Q}_2} \to \mathrm{Gal}_2(\mathbb{F}_2)$, which is semisimple but sends a Frobenius element to $\begin{bmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix}$ (a unipotent element of $\mathrm{Gal}_2(\mathbb{F}_2)$).

It is also not helpful to consider embeddings ${}^LG \hookrightarrow \mathrm{GL}_d$, because the property of being a completely reducible subgroup is *not well-behaved undering embeddings* for disconnected groups. For example, consider

$$\mathbb{Z}/3 \to \mathrm{GL}_2(\mathbb{F}_3)$$
$$\bar{1} \mapsto \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}$$

$\mathbb{Z}/3 \subset \mathbb{Z}/3$ is a completely reducible subgroup (because it is the full group), but its image in $\mathrm{GL}_2(\mathbb{F}_3)$ is not completely reducible. As a consequence, even proving $\bar{\rho}$ is tamely ramified is not straightforward.

We managed to reduce Theorem A to the linear algebra problem (see below). The author does not know of an elementary proof of Theorem A for low rank ramified unitary groups.

A linear algebra problem A familiar fact from linear algebra is that two diagonalizable matrices X and Y are simultaneously diagonalizable if and only if they commute with each other, that is, $XY = YX$.

We can reinterpret the simultaneous diagonalization theorem from the perspective of algebraic groups. First of all, the center of GL_d does not play a role in a diagonalization problem, so we can assume X, Y as elements of SL_d. Matrices X and Y correspond to inner automorphisms $\mathrm{Int}(X)$, $\mathrm{Int}(Y)$ of the algebraic group SL_d. Now, X is a diagonalizable matrix if and only if there exists a Borel $B \subset \mathrm{SL}_d$ and a maximal torus $T \subset B$ such that the Borel pair (B, T) is $\mathrm{Int}(X)$-stable. Two matrices X and Y are simultaneously diagonalizable if and only if there exists a Borel pair (B, T) of SL_d fixed by both $\mathrm{Int}(X)$ and $\mathrm{Int}(Y)$. The simultaneous diagonalization theorem follows from the following standard linear algebraic group fact: two commuting semisimple elements are contained in a maximal torus ([St68, Corollary 8.6]).

In this paper, we need a generalized simultaneous diagonalization problem. Instead of inner automorphisms, we allow for outer automorphisms; and instead of the commuting relation, we only impose the metacyclic relation; finally, we relax the semisimplicity assumption. We prove the following:

Theorem C. *(Corollary 2.3.7) Let H be a connected reductive group over an algebraically closed field and let τ, σ be two automorphisms of H. Assume*

(MC) $\sigma\tau\sigma^{-1} = \tau^q$ *for some integer q,*

(WS) τ *is a semisimple automorphism and $\langle\sigma, \tau\rangle$ is a pseudo-completely reducible subgroup of* $\mathrm{Aut}(H)$.

Then there exists a Borel pair (B, T) of H such that T is both σ- and τ-stable while B is τ-stable.

Here (MC) stands for metacyclicity and (WS) stands for weak semisimplicity. We remark that by forming the semi-direct product $H' := H \rtimes \langle\sigma, \tau\rangle$, we can instead insist that both σ and τ are inner automorphisms of H' while allowing H' to be a disconnected reductive group.

1.5. A tame inertial local Langlands correspondence The second application of the factorization theorem (Theorem A) is tame inertial Local Langlands Correspondence. Recall that $\underline{G}$ denotes the reductive quotient of a superspecial parahoric of G. An *inertial Deligne-Lusztig datum* is a pair $(\underline{S}, \underline{\chi})$ where $\underline{S}$ is a maximal torus of $\underline{G}$ and $\underline{\chi}$ is a character $\underline{S}(\kappa_F) \to \bar{\mathbb{F}}_p^\times$. Computation shows that geometric conjugacy classes of inertial Deligne-Lusztig data and of tame inertial types $I_F \to {}^L G(\bar{\mathbb{F}}_p)$ admit the same combinatorial parametrization, and thus there exists a set-theoretic bijection between these two. However, such a bijection depends a priori on certain choices and its arithmetic significance is not clear. Using the Langlands-Shelstad factorization, we give a theoretic explanation of this bijection and show it is compatible with (indeed determined by) the LLC for algebraic tori.

Theorem D. *(Corollary 4.5.5) There exists a natural bijection*

$$\{\textit{Inertial Deligne-Lusztig data } (\underline{S}, \underline{\chi})\}_{/\underline{G}(\bar{\mathbb{F}}_p)} \xrightarrow{\cong} \{\textit{Tame inertial types } I_F \to {}^L G(\bar{\mathbb{F}}_p)\}_{/\widehat{G}(\bar{\mathbb{F}}_p)}$$

The theorem above is a natural extension of the Deligne-Lusztg duality for finite groups of Lie type to quasi-split tame p-adic groups, and we call it *the Deligne-Lusztig correspondence.* It is the starting point of the generalization of the Gee-Herzig-Savitt recipe for Serre weights for unramified groups to tame groups; we will elaborate on this topic in the next subsection.

1.6. Explicit Serre weight conjectures, after Herzig In this subsection, we present a generalization of Herzig's explicit Serre weight recipe.

By the non-abelian Shapiro's lemma ([L23, Proposition 1]), working with the F-group G is equivalent to working with the $\mathbb{Q}_p$-group $\operatorname{Res}_{F/\mathbb{Q}_p} G$. Assume G is a quasi-split tame group over $\mathbb{Q}_p$ in the rest of the introduction for simplicity of notation, without loss of generality. Recall that $\mathcal{G}^\circ$ is a superspecial parahoric of G, and $\underline{G}$ is the reductive quotient of $\mathcal{G}^\circ$.

For technical simplicity, we assume both G and $\underline{G}$ admit a local twisting element and $\underline{G}$ has a simply-connected derived subgroup. Since G is the generic fiber of $\mathcal{G}^\circ$, we denote by $\eta_{\mathbb{Q}_p}$ a local twisting element for G; and since $\underline{G}$ is the reductive part of the special fiber of $\mathcal{G}^\circ$, we denote by $\eta_{\mathbb{F}_p}$ a local twisting element for $\underline{G}$. By restricting to the maximal unramified subtorus of a maximal torus of G, we can regard $\eta_{\mathbb{Q}_p}$ as an element of the character lattice of $\underline{G}$.

For each tame inertial L-parameter $\tau : I_{\mathbb{Q}_p} \to {}^LG(\bar{\mathbb{F}}_p)$, we define

$$W^?(\tau) := \mathcal{R}(\mathrm{JH}(\bar{V}(\mathrm{DL}^{-1}(\tau)) \otimes W(w_0(\eta_{\mathbb{F}_p} - \eta_{\mathbb{Q}_p})))).$$

Here $\mathcal{R}$ is Herzig's involution operator and $\bar{V}$ is the reduction mod p of the Deligne-Lusztig induction functor (with a sign modification). See Subsection 6.3 for other unfamiliar notations and clarifications.

Remark

- Our definition is inspired by the work of [LLHLM22] for $\operatorname{Res}_{E/\mathbb{Q}_p} \mathrm{GL}_d$.
- When G is unramified, $\eta_{\mathbb{Q}_p} - \eta_{\mathbb{F}_p} = 0$ and thus $W^?(\tau)$ recovers the definition found in [GHS, Section 9].
- $\mathrm{DL}^{-1}(\tau)$ is only well-defined up to geometric conjugacy while the input for the Deligne-Lusztig induction needs to be well-defined up to rational conjugacy. As a consequence, the composition $\bar{V} \circ \mathrm{DL}^{-1}$ is ambiguous (it is a multi-valued map).

L-packets We want to elaborate on the last bullet point of the remarks above. To resolve the ambiguity for $\bar{V} \circ \mathrm{DL}^{-1}$, [GHS] considers only the *maximally split* rational conjugacy class of DL^{-1}, which is unique after imposing certain technical assumptions on G, indeed, for sufficiently generic tame types τ, maximal splitness of $\mathrm{DL}^{-1}(\tau)$ is automatic ([Her09, Proposition 6.20, Lemma 6.24]), and therefore so is the uniqueness of the rational conjugacy class (under certain technical assumptions).

From the perspective of the Local Langlands Correspondence, an L-parameter should correspond to an L-packet of admissible representations, rather than a single admissible representation. When G is an unramified p-adic group, the maximally split rational conjugacy class of $\mathrm{DL}^{-1}(\tau)$ is expected to correspond to the *generic constituent* of the L-packet. For more general groups, there can be multiple generic constituents in L-packets. In generic situations, we expect the various rational conjugacy classes of $\mathrm{DL}^{-1}(\tau)$ to correspond to the various generic constituents in the corresponding L-packet.

The reader may want to interpret $W^?(\tau)$ not as a set, but rather as a packet of sets where $\mathrm{DL}^{-1}(\tau)$ ranges over all rational conjugacy classes.

1.7. The geometric Breuil-Mézard conjecture: the potentially crystalline case

We refer to [L23B, Section 2.4, 2.5, 2.6] for the definition of the potentially crystalline stacks $\mathcal{X}_{{}^LG}^{\mathrm{crys},\lambda,\tau}$ of Hodge type λ and tame inertial type τ. Roughly speaking, a Hodge type is a conjugacy class of cocharacters λ of $\widehat{G}$. Since cocharacters of $\widehat{G}$ are identified with characters of G, we regard λ as an element of the character lattice of G.

Denote by $V(\lambda-\eta_{\mathbb{Q}_p})$ the (restriction to $\mathcal{G}^\circ$ of the) irreducible algebraic representation of highest weight $\lambda-\eta_{\mathbb{Q}_p}$. For a finite-dimensional representation R of $\mathcal{G}^\circ$ over a finite extension of $\mathbb{Q}_p$, write $\bar{R}$ for the semisimplification of the reduction mod p of a $\mathcal{G}^\circ$-invariant lattice of R.

The natural generalization of [LLHLM23, Theorem 1.4.5(1)] is the following conjecture.

Conjecture 1. *If λ is regular dominant and τ is a sufficiently generic tame inertial type, then*

$$\mathcal{X}^{\mathrm{crys},\lambda,\tau}_{{}^LG,\mathrm{red}}=\bigcup_{\sigma\in\mathrm{JH}(\overline{\bar{V}(\mathrm{DL}^{-1}(\tau))\otimes V(\lambda-\eta_{\mathbb{Q}_p})})}\mathcal{C}_\sigma$$

Write $[\mathcal{X}^{\mathrm{crys},\lambda,\tau}_{{}^LG,\mathbb{F}_p}]$ for the cycle class of $\mathcal{X}^{\mathrm{crys},\lambda,\tau}_{{}^LG,\mathbb{F}_p}$ in the Chow group of $\mathcal{X}_{{}^LG,\mathrm{red}}$. The conjecture above has the following refinement generalizing [EG23, Conjecture 8.2.2].

Conjecture 2. *For each parahoric Serre weight σ, there exists an effective top-dimensional cycle $\mathcal{Z}_\sigma$ on $\mathcal{X}_{{}^LG,\mathrm{red}}$ such that*

$$[\mathcal{X}^{\mathrm{crys},\lambda,\tau}_{{}^LG,\mathbb{F}_p}]=\sum_\sigma[\overline{\bar{V}(\mathrm{DL}^{-1}(\tau))\otimes V(\lambda-\eta_{\mathbb{Q}_p})}:\sigma]\mathcal{Z}_\sigma$$

for all regular λ and tame inertial types τ.

We don't have evidence for the conjecture except that it agrees with the ramified general linear case $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_n$ ([EG23]) and the split group case ([FLH]). However, since the method of [FLH] is purely local and group-theoretical, and it seems plausible that the arguments of [FLH] generalize to general tame groups once the corresponding players are correctly defined.

Finally, we explain the construction of $\sigma\mapsto\mathcal{C}(\sigma)$. There exist natural bijections between the following objects:

(1) equivalence classes of regular parahoric Serre weights,
(2) geometric conjugacy classes of regular based inertial Deligne-Lusztig data of niveau 1,
(3) $\widehat{G}(\bar{\mathbb{F}}_p)$-conjugacy classes of regular inertial L-parameters $I_F\to{}^LB(\bar{\mathbb{F}}_p)$ where ${}^LB=\widehat{B}\rtimes\mathrm{Gal}(L/F)$ is a Borel of ${}^LG$.

The bijection (1) $\Leftrightarrow$ (2) follows from Proposition 6.2.5; and the bijection (2) $\Leftrightarrow$ (3) follows from Theorem D. So, we identify (1), (2) and (3) implicitly in the rest of this subsection. A regular parahoric Serre weight σ of niveau 1 admits Herzig's presentation $(1,\mu_\sigma)$ (see 6.2.1).

Definition 1. *$\mathcal{C}(\sigma)$ is defined to be the closure of $\bar{\mathbb{F}}_p$-points of $\mathcal{X}_{{}^LG,\mathrm{red}}$ corresponding to L-parameters $\bar{\rho}:\mathrm{Gal}_{\mathbb{Q}_p}\to{}^LG(\bar{\mathbb{F}}_p)$ that factor through a unique Borel ${}^LB$ such that $\bar{\rho}|_{I_{\mathbb{Q}_p}}:I_{\mathbb{Q}_p}\to{}^LB(\bar{\mathbb{F}}_p)$ has Herzig's presentation $(1,-w_0(\mu_\sigma)-\eta_{\mathbb{F}_p})$.*

The reader can verify that our definition is consistent with [EG23, Definition 5.5.1] for $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_n$.

1.8. Future directions Before we finish the introduction, we raise the following natural questions that are not addressed in this paper.

Question A How about wildly ramified p-adic groups?

If $p\geq 5$, then any connected reductive group over a p-adic field F is isogenous to a product of groups of the form $\mathrm{Res}_{K/F}G$ where G is tame over K. The Weil restricted case follows from the tame case by non-abelian Shapiro's lemma (see, for example, [L23, Section 7.2]). So it remains to consider the $p = 2, 3$-case. It is probably approachable by a case-by-case analysis, and we expect the most complicated case to be related to the triality.

Question B How about the mod l ($l \neq p$) analogue?

To study mod l Langlands parameters, we need a further generalization of Theorem C, which allows τ to be a non-semisimple automorphism. We expect generalizations of Theorem B-D to hold under some mild assumptions on the prime p.

Question C How about crystalline lifts of semisimple mod p Langlands parameters? Can we formulate a conjecture relating crystalline lifts and Serre weights?

When G is a split group, crystalline lifts of semisimple L-parameters are constructed in [L22]. When G is ramified, crystalline L-parameters do not exist. So it is an interesting question whether crystalline lifts exist when G is unramified and non-split.

The reason that crystalline L-parameters do not exist for ramified groups is probably simply because the naïve definition of crystallinity is incorrect. Similar issues arise when people study the Satake correspondence for ramified groups, in which case unramified L-parameters do not exist; nonetheless, we can still define "spherical" L-parameters for ramified groups (see [Zhu15]). Similarly, we say a p-adic L-parameter is "crystalline" if it is potentially crystalline and its corresponding Weil-Deligne representation is "spherical" in the sense of [Zhu15, Definition 6.3]. It seems an interesting question to explore analogues of the crystalline lift aspects of the Serre weight conjectures.

1.8.1. Acknowledgements The author thanks Bao Le Hung for many inspiring conversations. The author is grateful to Tasho Kaletha for helpful correspondence and sharing the manuscript of his book. The author thanks Florian Herzig for his interest in this project and for his correspondence. Many ideas in this paper originate from the author's thesis research and the author thanks his PhD supervisor David Savitt for introducing the subject to the author. Finally, the reader will find the influence of the work of Emerton-Gee, Gee-Herzig-Savitt and Le-Le Hung-Levin-Morra to the author.

1.8.2. Notation and conventions Write $\breve{F}$ for the maximal unramified extension of F inside a fixed separable closure F^s. For each finite extension E of F, denote by κ_E the residue field of E, and denote by $\mathcal{O}_E$ the ring of integers of E.

We denote $- \otimes_{\mathbb{Z}} -$ by $- \otimes -$, and denote $\mathrm{Hom}_{\mathrm{Grp}}(-,-)$ by $\mathrm{Hom}(-,-)$.

Write $\mathbb{Q}_{p'}/\mathbb{Z}$ for the prime-to-p divisible subgroup of $\mathbb{Q}/\mathbb{Z}$. Note that $\mathbb{Q}_{p'}/\mathbb{Z}$ is isomorphic to $\bar{\mathbb{F}}_p^\times$ as an abelian group.

2. Metacyclic actions on reductive groups

Contents

2.1. Extending pseudo-parabolics of disconnected groups

2.1.1. Dynamic methods We recall the definitions in [L22, 2.3]. Let H be an algebraic group over a ring k. Let $f : \mathbb{G}_m \to H$ be a *scheme morphism.* Define the following functor on the category of k-algebras

$$P_H(f)(A) = \{g \in H(A)| \lim_{t\to 0} f(t)gf(t)^{-1} \text{ exists.}\}$$

where A is a general k-algebra. We call f a *fake cocharacter*. Here "a limit exists" means the scheme morphism $\mathbb{G}_m \to H$, defined by $t \mapsto f(t)gf(t)^{-1}$, extends to a scheme morphism $\mathbb{A}^1 \to H$. Note that $P_H(f)$ is not representable in general. We define similarly $U_H(f)$ by setting $A \mapsto \{g \in H(A)| \lim_{t\to 0} f(t)gf(t)^{-1} = 1\}$, and $Z_H(f)$ by setting $A \mapsto \{g \in H(A)|fg = gf\}$.

2.1.2. Definition Let H be an algebraic group over an algebraically closed field. A fake cocharacter $f : \mathbb{G}_m \to H$ is said to be *relevant* if the functor $P_H(f)$ is representable by a linear algebraic group. In particular, all maximal tori of $P_H(f)$ are conjugate to each other, by an element of $P_H(f)$.

The reason we introduce the notion of fake cocharacters is because of the following powerful tool:

2.1.3. Lemma Let $M \hookrightarrow G$ be two possibly disconnected algebraic groups over an algebraically closed field k. Write M° for the neutral component of M. Let $\lambda, \mu : \mathbb{G}_m \to M^\circ$ be two relevant cocharacters of M°. Assume $P_{M^\circ}(\lambda) = P_{M^\circ}(\mu) =: P$.
(1) There exists a relevant cocharacter $f : \mathbb{G}_m \to M^\circ$ such that $P_G(f) = P_G(\mu\lambda)$ as a functor. In particular, the fake cocharacter $\mu\lambda : \mathbb{G}_m \to M^\circ$ is relevant.

We have $P_M(\lambda) \cap P_M(\mu) \subset P_M(\mu\lambda) = P_M(f)$.

Moreover, if M° is a reductive group, then $P_{M^\circ}(\lambda) = P_{M^\circ}(\mu\lambda)$.
(2) The limit

$$\lim_{t\to 0} \lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}$$

exists in the sense of subsection 2.1.1, and lies in P.
(3) Let u be an element of P. The limit

$$\lim_{t\to 0} \lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}\mu(t)^{-1}$$

exists in the sense of subsection 2.1.1 and lies in P.
(4) Now assume λ is a product of cocharacters $\lambda_1, \ldots, \lambda_s$ such that $P_{M^\circ}(\lambda_i) = P$ for all i. (1), (2) and (3) remain true (for example, $P_G(\mu\lambda_1 \ldots \lambda_s) = P_G(f)$ for some cocharacter $f : \mathbb{G}_m \to M^\circ$).

Proof. Since all maximal tori in P are conjugate to each other and the image of a cocharacter is contained in a maximal torus, there exists an element $x \in P$ such that $(x\lambda x^{-1})\mu = \mu(x\lambda x^{-1})$. Write ξ for $x\lambda x^{-1}$.

(1) We have

$$\begin{aligned}\lim_{t\to 0} \mu(t)\lambda(t)g\lambda(t)^{-1}\mu(t)^{-1} &= \lim_{t\to 0} \mu(t)x^{-1}\xi(t)xgx^{-1}\xi(t)^{-1}x\mu(t)^{-1} \\ &= \lim_{t\to 0}(\mu(t)x^{-1}\mu(t)^{-1}) \cdot (\mu(t)\xi(t)xgx^{-1}\xi(t)^{-1}\mu(t)^{-1}) \cdot (\mu(t)x\mu(t)^{-1}) \\ &= \lim_{t\to 0} \mu(t)x^{-1}\mu(t)^{-1} \cdot \lim_{t\to 0} \mu(t)\xi(t)xgx^{-1}\xi(t)^{-1}\mu(t)^{-1} \cdot \lim_{t\to 0} \mu(t)x\mu(t)^{-1}\end{aligned}$$

Since $x \in P$, $\lim_{t\to 0} \mu(t)x\mu(t)^{-1}$ exists. So we have $P_G(\mu\lambda) = x^{-1}P_G(\mu\xi)x = P_G(x^{-1}\mu\xi x)$. Note that $f := x^{-1}\mu\xi x$ is a cocharacter.

It is obvious that $P_M(\lambda) \cap P_M(\mu) \subset P_M(\mu\lambda) = P_M(f)$.

Next we consider the "moreover" part. Since $\mu\xi = \xi\mu$, we can regard μ and ξ as elements in a cocharacter lattice $X_*(M^\circ, T)$ where T is a maximal torus containing both μ and ξ. Since $P_{M^\circ}(\mu) = P_{M^\circ}(\lambda) = P_{M^\circ}(\xi)$, μ and ξ lie in the same Weyl chamber (the Borel case) or the same wall of Weyl chamber (the non-Borel case). The cocharacter $\mu\xi$ is the sum of μ and ξ in the cocharacter lattice $X_*(G,T)$, and lies in the same (wall of) Weyl chamber. So $P_{M^\circ}(\mu\xi) = P_{M^\circ}(\mu) = P_{M^\circ}(\lambda)$. Since $x \in P$, we have $P_G(\mu\lambda) = x^{-1}P_G(\mu\xi)x = P_G(\mu) = P_G(\lambda)$.

(2) We have

$$\begin{aligned}\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1} &= x^{-1}\xi(t)x\mu(t)x^{-1}\xi(t)^{-1}x\mu(t)^{-1}\\ &= x^{-1}\cdot(\xi(t)x\xi(t)^{-1})\cdot(\xi(t)\mu(t)x^{-1}\mu(t)^{-1}\xi(t)^{-1})\cdot(\mu(t)x\mu(t)^{-1})\end{aligned}$$

By (1), $P = P_{M^\circ}(\lambda)\cap P_{M^\circ}(\mu) \subset P_{M^\circ}(\xi\mu)$, and thus the limits

$$\lim_{t\to 0}\xi(t)g\xi(t)^{-1},$$

$$\lim_{t\to 0}\xi(t)\mu(t)g^{-1}\mu(t)^{-1}\xi(t)^{-1}, \text{ and}$$

$$\lim_{t\to 0}\mu(t)g\mu(t)^{-1}$$

all exist and lies in P. As a consequence, $\lim_{t\to 0}\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1} \in P$.

(3) We have

$$\lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}\mu(t)^{-1} = (\lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}u\mu(t)^{-1}u^{-1})(u\mu(t)u^{-1}\mu(t)^{-1}).$$

So (3) follows from (2).

(4) The method is the same but notations are more complicated. The reader can consult [L22, Lemma 2.4] for a proof. □

2.1.4. Theorem (The Parabolic Extension Theorem) Let M be a possibly disconnected linear algebraic group over an algebraically closed field with reductive neutral component M°. Let H be a disconnected linear algebraic group which is a group extension

$$1 \to M \to H \to \langle\bar\gamma\rangle \to 1,$$

where $\langle\bar\gamma\rangle$ is a cyclic group $\mathbb{Z}/d$, regarded as a constant group scheme.

Let $f : \mathbb{G}_m \to M^\circ$ be a cocharacter such that

(PE1) $P_M(f)$ is stabilized by γ for some $\gamma \in H$ which lifts $\bar\gamma$;
(PE2) The normalizer of $P_M(f)$ in M is $P_M(f)$ itself.

Then there exists a cocharacter $f' : \mathbb{G}_m \to M^\circ$ such that

(1) $P_{M^\circ}(f) = P_{M^\circ}(f')$,
(2) $P_M(f) \subset P_M(f')$, and
(3) $\gamma \in P_H(f')$.

Proof. We have $P_{M^\circ}(\gamma f\gamma^{-1}) = \gamma P_{M^\circ}(f)\gamma^{-1} = P_{M^\circ}(f)$ by unravelling the definitions. Write $F(-)$ for $\gamma\cdot(-)\cdot\gamma^{-1}$ (conjugation-by-γ). Define

$$\mu := F^{d-1}(f)F^{d-2}(f)\dots F(f)f.$$

Say $\gamma^d = u \in M$. By (PE1), u is in the normalizer of $P_M(f)$ in M. By (PE2), $u \in P_M(f)$. Note that

$$F(\mu) = (ufu^{-1})\mu f^{-1}.$$

We have

$$\begin{aligned}
\lim_{t\to 0}\mu(t)\gamma\mu(t)^{-1} &= \lim_{t\to 0}\mu(t)\gamma\mu(t)^{-1}\gamma^{-1}\gamma \\
&= \lim_{t\to 0}\mu(t)F(\mu)(t)^{-1}\gamma \\
&= \lim_{t\to 0}\mu(t)f(t)\mu(t)^{-1}uf(t)^{-1}u^{-1}\gamma \\
&= \lim_{t\to 0}(\mu(t)f(t)\mu(t)^{-1}f(t)^{-1})(f(t)uf(t)^{-1})u^{-1}\gamma \\
&= \lim_{t\to 0}\mu(t)f(t)\mu(t)^{-1}f(t)^{-1}\cdot\lim_{t\to 0}f(t)uf(t)^{-1}\cdot u^{-1}\gamma
\end{aligned}$$

By Lemma 2.1.3, the limit $\lim_{t\to 0}\mu(t)f(t)\mu(t)^{-1}f(t)^{-1}$ exists and lies in $P_{M^\circ}(f)$. Since $u\in P_M(f)$, the limit $\lim_{t\to 0}f(t)uf(t)^{-1}$ exists and lies in $P_M(f)$. Therefore the limit $\lim_{t\to 0}\mu(t)\gamma\mu(t)^{-1}$ exists and lies in $P_M(f)\gamma$. So $\gamma\in P_H(\mu)$. Since $P_M(F^i(f)) = P_M(f)$ for all $i\geq 0$, we have $P_M(f) = \cap_{i\geq 0}P_M(F^i(f))\subset P_M(\mu)$. We've seen the fake cocharacter μ satisfies all the requirements. By the previous lemma, there exists a genuine cocharacter $f':\mathbb{G}_m\to M^\circ$ such that $P_H(\mu)=P_H(f')$. □

2.1.5. Corollary Let M be a linear algebraic group over an algebraically closed field such that

- the neutral component M° is reductive and
- the component group $\pi := M/M^\circ$ is a finite solvable group.

Let $\Pi\subset M$ be a solvable subgroup which maps surjectively onto π. For each Π-stable parabolic $P\subset M^\circ$, there exists a cocharacter $f:\mathbb{G}_m\to M^\circ$ such that

- $P = P_{M^\circ}(f)$, and
- $\Pi\subset P_M(f)$.

Proof. Let $\Pi_0\subset\Pi$ be a normal subgroup such that Π/Π_0 is cyclic. Write π_0 for the image of Π_0 in Π. Choose $\gamma\in\Pi$ which maps surjectively onto π/π_0.

Write $M^\circ\Pi_0$ for the subgroup of M generated by M° and Π_0.

By induction, there exists a cocharacter $f_0:\mathbb{G}_m\to M^\circ\Pi_0$ such that

- $P_{M^\circ}(f_0) = P$, and
- $\Pi_0\subset P_{M^\circ\Pi_0}(f_0)$.

Since Π_0 maps surjectively onto $(M^\circ\Pi_0)/M^\circ$, we have $P_{M^\circ\Pi_0}(f_0) = P\Pi_0$. Thus $P_{M^\circ\Pi_0}(f_0)$ is stabilized by Π; and the assumption (PE1) in the previous theorem is verified. Suppose $gh\in M^\circ\Pi_0$ lies in the normalizer of $P_{M^\circ\Pi}(f_0)$, and that $g\in M^\circ$ and $h\in\Pi_0$. It follows that g also stabilizes $P_{M^\circ\Pi}(f_0)\supset P$. So $g\in P$. Thus the normalizer of $P_{M^\circ\Pi_0}(f_0)$ in $M^\circ\Pi_0$ is contained in $P\Pi_0 = P_{M^\circ\Pi_0}(f_0)$; and we've verified the assumption (PE2). By the theorem above, there exists a cocharacter $f:\mathbb{G}_m\to M^\circ$ such that

- $P_{M^\circ} = P$,
- $P_{M^\circ\Pi_0}(f_0)\subset P_{M^\circ\Pi_0}(f)$, and
- $\gamma\in P_M(f)$.

The second and the third bullet point jointly imply $\Pi\subset P_M(f)$. □

2.2. Complete reducibility for disconnected groups We generalize Serre's notion of G-complete reducibility to disconnected groups.

2.2.1. Definition Let H be a linear algebraic group scheme. A subgroup of H is said to be *a pseudo-parabolic* if it is of the form $P_H(f)$ for some cocharacter $f : \mathbb{G}_m \to H^\circ$.

A subgroup of H is said to be *a pseudo-Levi* if it is of the form $Z_H(f)$ for some cocharacter $f : \mathbb{G}_m \to H^\circ$.

2.2.2. Remark The terminology is misleading. If we only consider groups H over a characteristic 0 field, then the notion of "pseudo-parabolic" is strictly stronger than the usual notion of parabolic (closed subgroups P such that G/P is a projective variety).

Here is a dumb example: let H be $\mathrm{GL}_2 \rtimes (\{1, \sigma\} \times \{1, \tau\})$ where both σ and τ acts on GL_2 by $(-) \mapsto (-)^{-t}$; any pseudo-parabolic that contains σ must contain τ as well. So all pseudo-parabolics are conjugate to either B or $B \rtimes (\{1, \sigma\} \times \{1, \tau\})$ where B is the subgroup of upper-triangular matrices. However, the group $B \rtimes \{1, \sigma\}$ is a parabolic subgroup in the usual sense.

2.2.3. Definition Let H be an algebraic group over an algebraically closed field. A *pseudo-parabolic* P of an algebraic group H is said to be a *proper* pseudo-parabolic if $P \cap H^\circ \neq H^\circ$.

An abstract subgroup Γ of H is said to be *pseudo-irreducible* in H if Γ is not contained in any proper pseudo-parabolic subgroup.

An abstract subgroup Γ of H is said to be *pseudo-completely reducible* in H if whenever Γ is contained in a proper pseudo-parabolic, it is also contained in a correponding pseudo-Levi.

2.2.4. Remark The notion of pseudo-irreducibility is well-behaved in the following situation: let M be a subgroup of H and assume $\Gamma \subset M$; if Γ is pseudo-irreducible in H then Γ is also pseudo-irreducible in M.

This is not immediate from the definitions. Suppose $H = \mathrm{GL}_{2n} \rtimes \{1, \sigma\}$. and let $M = \mathrm{SO}_{2n} \rtimes \{1, \sigma\} \subset H$. The above paragraph claims all σ-stable proper parabolics of SO_{2n} extend to a parabolic of GL_{2n} which is σ-stable. The Parabolic Extension Theorem 2.1.4 is where we comfirmed the above claim in full generality.

2.2.5. Definition Let M be a linear algebraic group over an algebraically closed field. A pseudo-parabolic P of M is said to be *big* if P maps surjectively onto M/M°.

2.2.6. Lemma Let M be a disconnected linear algebraic group over an algebraically closed field, with reductive neutral component. Assume M/M° is solvable. A subgroup P of M is a big pseudo-parabolic if and only if

- $P \cap M^\circ$ is a parabolic of M, and
- P maps surjectively onto M/M°.

Proof. By induction, we can assume M/M° is cyclic. Choose an element $\gamma \in P$ which maps to a generator of M/M°. By Corollary 2.1.5, there exists a cocharacter $f : \mathbb{G}_m \to M^\circ$ such that

- $P \cap M^\circ = P_{M^\circ}(f)$, and
- $\gamma \in P_M(f)$.

So $P \subset P_M(f)$. Since P is big, we must have $P = P_M(f)$. $\square$

2.2.7. Lemma Let M be a linear algebraic group over an algebraically closed field, with reductive neutral component. Assume M/M° is solvable. Two big pseudo-parabolics P, Q of M are the same if and only if $P^\circ = Q^\circ$.

Proof. Suppose $P^\circ = Q^\circ$. By induction, we can assume M/M° is cyclic. Let $g \in P$ and $h \in Q$ be elements in the same component of M which generate all components. So $h^{-1}g \in N_M(P^\circ) \cap M^\circ = N_{M^\circ}(P^\circ) = P^\circ$. Thus $P = Q$. □

2.2.8. Remark Both of the lemmas above fail without the bigness assumption. Let ω be a primitive cubic root of unity. Let H be $\mathrm{GL}_2 \rtimes (\{1, \sigma, \sigma^2\} \times \{1, \tau, \tau^2\})$ where both σ and τ acts on GL_2 by conjugation by $\begin{bmatrix} \omega & \\ & \omega^{-1} \end{bmatrix}$ and $\frac{1}{2}\begin{bmatrix} -1 & \omega^{-1} - \omega \\ \omega^{-1} - \omega & -1 \end{bmatrix} =: x$, respectively. The group H contains no big pseudo-parabolic. The two pseudo-parabolics $B \rtimes \langle \sigma \rangle$ and $B \rtimes \langle x^{-1}\tau \rangle$ are not conjugate to each other.

At the end of this section, we clarify the relation between various notions of semisimplicity.

We recall Steinberg's definition ([St68, Section 9]) of quasi-semisimple automorphisms:

2.2.9. Definition An automorphism $f : M \to M$ of connected linear algebraic groups is said to be *quasi-semisimple* if there exists a Borel $B \subset M$ and a maximal torus $T \subset B$ such that $f(B) = B$ and $f(T) = T$.

A well-known theorem is

2.2.10. Theorem ([St68, Theorem 7.5]) Semisimple automorphisms are quasi-semisimple.

2.2.11. Definition An automorphism $f : M \to M$ of connected linear algebraic groups is said to be *pseudo-semisimple* if $\langle f \rangle$ is a pseudo-completely reducible subgroup in $\mathrm{Aut}(M)$.

2.2.12. Proposition (1) Semisimple automorphisms of reductive groups are pseudo-semisimple.

(2) Pseudo-semisimple automorphisms of reductive groups are quasi-semisimple.

(3) A quasi-semisimple automorphism is semisimple if and only if its class in $\mathrm{Aut}(G)/\mathrm{Int}(G)$ has order prime to the characteristic (exponent) of the field.

Proof. (1) This is [L22, Proposition 2.2].

(2) Let $f : M \to M$ be an automorphism of reductive groups. By replacing M by $M/Z_M(M)$, we can assume M is centerless and $\mathrm{Aut}(M) = M \rtimes \mathrm{Out}(M)$ where $\mathrm{Out}(M)$ is a finite group.

Suppose $f : M \to M$ is pseudo-semisimple. By [St68, Theorem 7.2], there exists a Borel $B \subset M$ which is f-stable. By Corollary 2.1.5, there exists a pseudo-parabolic $P \subset \mathrm{Aut}(M)$ such that

- $P \cap M = B$,
- $f \in P$.

Since $\langle f \rangle$ is pseudo-completely reducible in $\mathrm{Aut}(M)$ and $\langle f \rangle$ lies in the pseudo-parabolic P, we can choose a cocharacter $\lambda : \mathbb{G}_m \to B$ suh that the pseudo-Levi $Z_{\mathrm{Aut}(M)}(\lambda)$ contains $\langle f \rangle$. The neutral component of $Z_{\mathrm{Aut}(M)}(\lambda)$ is a maximal torus of M, and is stable under f.

(3) See the remarks at the beginning of [St68, Section 9]. □

2.3. Simultaneous diagonalization of metacyclic actions on reductive groups

In this section, we study when two automorphisms of a reductive group fix a common maximal torus.

2.3.1. Remark Instead of thinking about outer automorphisms of connected groups, it is helpful to form a semi-direct product and think about inner automorphisms of disconnected linear algebraic groups for two reasons:

- the powerful machinery of pseudo-parabolics are available, and
- the framework of disconnected groups is more flexible and allows us to maneuver to disconnected groups that are not semi-direct products.

Since outer automorphism groups of semisimple connected groups are finite, we don't lose much by allowing only considering disconnected groups of finite type.

It is useful to pass to the adjoint quotient.

2.3.2. Lemma (1) Let H be a possibly disconnected algebraic group. Let $\pi : H' \to H$ be a homormorphism such that $H'^{\circ} \to H^{\circ}$ is a central isogeny. If $P_H(f)$ is a pseudo-parabolic subgroup of H, then there exists a pseudo-parabolic subgroup $P_{H'}(f')$ of H' such that $\pi(P_{H'}(f')) = P_H(f)$.

(2) A subgroup Γ of H is pseudo-completely reducible in H if and only if it is pseudo-completely reducible in the adjoint quotient of H.

Proof. (1) Let T' be a maximal torus of H'°. Then $T = \pi(T')$ is a maximal torus of H°. We can assume f is valued in T. The homomorphism of cocharacter groups $X_*(T') \to X_*(T)$ has finite cokernel of cardinality d. The cocharacter f^d admits a lift $f' : \mathbb{G}_m \to T'$. We have $\pi(P_{H'}(f')) = P_H(f^d) = P_H(f)$.

(2) It is an immediate consequence of (1). □

2.3.3. Definition Let $\gamma \in \mathrm{Aut}(H)$ where H is a connected algebraic group. We say γ acts *innerly* on H if γ is an inner automorphism, and we say γ acts *semisimply* on H if γ is a semisimple automorphism.

2.3.4. Lemma Let Γ be a pseudo-completely reducible subgroup of H, and let γ be a element of Γ which acts on H° innerly by conjugation. Assume

- H° is reductive,
- Γ is solvable, and
- $\langle\gamma\rangle$ is a normal subgroup of Γ.

Then γ acts on H° semisimply by conjugation.

Proof. By passing to the adjoint quotient, we can assume H° is centerless. Say $\gamma(g) = hgh^{-1}$ for all $g \in H^{\circ}$. Since H° is centerless, h is unique. So for each $\sigma \in \Gamma$, $\sigma(h) = h^a$ for some a. Write $h = h_s h_u$ where h_s is the semisimple part and h_u is the unipotent part (the multiplicative Jordan decomposition). For each $\sigma \in \Gamma$, by [Sp98, Theorem 2.4.8], $\sigma(h_u) = \sigma(h)_u = (h^a)_u = h_u^a$ for some integer a. So $\langle h_u\rangle$ is stable under Γ. By [BT71, Corollaire 3.9], there exists a parabolic subgroup P of H° such that

- $h_u \in R_u(P)$ (the unipotent radical of P),
- $N_{H^{\circ}}(\langle h_u\rangle) \subset P$, and
- Γ fixes P.

By Corollary 2.1.5, there exists a cocharacter $f : \mathbb{G}_m \to H^{\circ}$ such that

- $P = P_{H^{\circ}}(f)$, and
- $\Gamma \subset P_H(f)$.

Since Γ is pseudo-completely reducible in H, we can choose f so that $\Gamma \subset Z_H(f)$. Since $h = h_s h_u \in Z_H(f)$, we have $h_u \in Z_H(f)$ by [Sp98, Theorem 2.4.8]. Therefore $h_u \in Z_H(f) \cap R_u(P) = \{1\}$. So $h = h_s$ is a semisimple element. □

2.3.5. Theorem Let M be a possibly disconnected algebraic group over an algebraically closed field with reductive neutral component M°. Let $\langle\tau\rangle \rtimes \langle\sigma\rangle \subset M$ be a metacyclic group subgroup. Assume τ acts on M° semisimply. We have either

(I) the neutral component of $(M^\circ)^\tau := \{x \in M^\circ | \tau(x) = x\}$ is a torus, or
(NI) The subgroup $\langle\tau\rangle \rtimes \langle\sigma\rangle \subset M$ is not pseudo-irreducible in M.

Proof. By replacing M by the subgroup generated by $\langle\tau\rangle \rtimes \langle\sigma\rangle$ and the derived subgroup of the neutral component $[M^\circ, M^\circ]$ we can and do assume M° is semi-simple.

Let $H = (M^\circ)^\tau$ be the fixed point subgroup. Since M° is semi-simple and τ is semi-simple, H° is a reductive group by [St68, Corollary 9.4].

It is clear that $\sigma(H^\circ) = H^\circ$. By [St68, Theorem 7.2], there exists a Borel $B \subset H^\circ$ which is σ-stable. Since τ acts trivially on B, the group B is $\langle\tau\rangle \rtimes \langle\sigma\rangle$-stable. Write H' for the subgroup of M generated by H° and $\langle\tau\rangle \rtimes \langle\sigma\rangle$. By Corollary 2.1.5, there exists a cocharacter $f : \mathbb{G}_m \to H^\circ$ such that

- $B = P_{H^\circ}(f)$, and
- $\gamma, \tau \in P_{H'}(f) \subset P_M(f)$.

The group $P_M(f)$ is a pseudo-parabolic subgroup of M. Since $P_M(f) \cap H^\circ = B$ and $H^\circ \subset M^\circ$, we have either $P_M(f) \cap M^\circ \neq M^\circ$ or $P_M(f) \cap H^\circ = H^\circ$ (negating both yields an easy contradiction). The first case implies the subgroup $\langle\tau\rangle \rtimes \langle\sigma\rangle$ is not pseudo-irreducible in M. The second case implies $H^\circ = B$. Since B is a Borel of H°, it forces H° to be a torus. $\square$

2.3.6. Lemma Let γ be a quasi-semisimple automorphism of a connected reductive group H over an algebraically closed field k. If $(H^\gamma)^\circ \subset Z_H(H)$ (the center of H), then H is also a torus.

Proof. Choose a Borel pair $T \subset B$ of H which is τ-stable. Let U be the unipotent radical of B. Assume U is a non-trivial group. Let β be the highest positive root of H with respect to B and T. The root group U_β is a characteristic subgroup of U, and is τ-stable. Denote by w_0 the longest Weyl group element with respect to T and B. It is clear that $\tau(w_0) = w_0$. The root group $U_{-\beta} = w_0 U_\beta w_0^{-1}$ is also τ-stable. Write H_β for the subgroup of H generated by U_β and $U_{-\beta}$. By the structure theory of reductive groups ([Sp98, Chapter 9]), there exists a surjective homomorphism $\psi : \mathrm{SL}_2 \to H_\beta$. So H_β is either SL_2 or PGL_2, and has no outer automorphisms. In particular, the fixed-point subgroup H_β^τ contains a maximal torus T_β of H_β. By the assumption $(H^\gamma)^\circ \subset Z_H(H)$, we have $T_\beta \subset (Z_H(H) \cap H_\beta)^\circ \subset Z_{H_\beta}(H_\beta)^\circ = \{1\}$, which is a contradiction. So U must be a trivial group, and $B = T$. $\square$

2.3.7. Corollary Let M be a possibly disconnected algebraic group whose neutral component is reductive. Let $\langle\tau\rangle \rtimes \langle\sigma\rangle \subset M$ be a pseudo-completely reducible metacyclic subgroup in M. If τ acts semisimply on M°, there exists a maximal torus T of M° which is $\langle\sigma, \tau\rangle$-stable, and a Borel $B \supset T$ which is τ-stable.

Proof. Assume without loss of generality that $\langle\sigma, \tau\rangle$ is pseudo-irreducible in M. By Theorem 2.3.5, $C = ((M^\circ)^\tau)^\circ$ is a torus. Write $L = Z_{M^\circ}(C)$, which is a Levi subgroup of M° stable under both σ and τ-action. We have $(L^\tau)^\circ = ((M^\circ)^\tau)^\circ \cap L = C \cap L = C$. By Lemma 2.3.6, L is a torus. Since τ is a semisimple automorphism, τ fixes a Borel pair $T \subset B$ of M°. Since T is abelian, we have $T \subset L$. Since T is a maximal torus, we have $T = L$. $\square$

2.4. Semisimple conjugacy classes in disconnected groups

The key tool is [KS99, Theorem 1.1.A]. We include a proof here for lack of reference in characteristic p.

2.4.1. Theorem Let M be a connected reductive group over a field. Let $\theta : M \to M$ be a quasi-semisimple automorphism. Write $M^{\theta\circ}$ for the neutral component of the fixed point group of θ in M. We have

(1) $M^{\theta\circ}$ is a reductive group,
(2) Let T^0 be a maximal torus of $M^{\theta\circ}$. Then $Z_M(T^0)$ is a maximal torus of M (which is clearly θ-stable). Moreover, for any Borel $B^0 \supset T^0$ of $M^{\theta\circ}$, there exists a θ-stable Borel B of M° containing $Z_M(T^0)$ and B^0.
(3) Conversely, let (B, T) be a θ-stable Borel pair of M. Then $(M^{\theta\circ} \cap B, M^{\theta\circ} \cap T)$ is a Borel pair of $M^{\theta\circ}$.
(4) If there is a θ-stable pinning $(M, B, T, \{u_\alpha\}_{\alpha \in R(B,T)})$ of M, then the map $N_{M^{\theta\circ}}(T^{\theta\circ})/T^{\theta\circ} \to (N_M(T)/T)^\theta$ is a bijection.

Proof. Part (1) and (3) can be found in [St68]. We first prove part (2). Define $H := Z_M(T^0)$, which is a reductive group since M is reductive. The fixed point group $H^{\theta\circ}$ is reductive by part (1), and $H^{\theta\circ} \subset M^{\theta\circ}$ (so T^0 is a maximal torus of $H^{\theta\circ}$). Since $H^{\theta\circ}$ is reductive and T^0 is a maximal torus thereof, $Z_{H^{\theta\circ}}(T^0) = T^0$. On the other hand, by definition, $Z_{H^{\theta\circ}}(T^0) = H^{\theta\circ}$. Thus $H^{\theta\circ} = T^0$. Lemma 2.3.6 applied to the group H shows H is a torus.

Next we prove the "moreover" part. There exists a θ-stable Borel pair (B, T). By part (3), $(B^1, T^1) := (B, T) \cap M^{\theta\circ}$ is a Borel pair of $M^{\theta\circ}$. Thus there exists $g \in M^{\theta\circ}$ such that $g(B^1, T^1)g^{-1} = (B^0, T^0)$. Since g commutes with θ, it is clear that gBg^{-1} is a θ-stable Borel.

Now consider part (4). Let α be a simple positive root of $R(B, T)$. The root group $u_{\alpha|_{T^0}} : t \mapsto \prod_{\beta|_{T^{\theta\circ}} = \alpha|_{T^{\theta\circ}}} u_\beta(t)$ is θ-stable of $T^{\theta\circ}$-weight $\alpha|_{T^{\theta\circ}}$. Since $u_{\alpha|_{T^0}}$ lies in the unipotent radical of B, we know it is a root group of $M^{\theta\circ}$, and $\alpha|_{T^{\theta\circ}} \neq 1$ is a simple positive root of $R(B^{\theta\circ}, T^{\theta\circ})$. In particular, $\alpha \in R(B, T)$ is a positive root if and only if $\alpha|_{T^{\theta\circ}}$ is a positive root. Thus the B is the unique Borel of M containing T and $B^{\theta\circ}$. Let $w \in (N_M(T)/T)^\theta$. There exists $w' \in N_{M^{\theta\circ}}(T^{\theta\circ})/T^{\theta\circ}$ such that $w^{-1}w' \in N_M(B)/T$ since there is a bijection between Weyl chambers of $M^{\theta\circ}$ and θ-stable Weyl chambers of M. Thus $w^{-1}w' \in T/T$, and $w = w'$. □

We remark that part (4) of Theorem 2.4.1 is also discussed in the paragraph after [KS99, Theorem 1.1.A]; and their argument works in characteristic p without change.

2.4.2. Definition In a disconnected algebraic group M, two elements x and y are said to be *conjugate to each other* if $x = gyg^{-1}$ for some $g \in M^\circ$.

2.4.3. Lemma Let M be a possibly disconnected reductive group over an algebraically closed field. There is a bijection

$$\{\text{Conjugacy classes of tuples } (s, T^0)\} \xrightarrow{(s,T^0)\mapsto s} \{\text{Semisimple conjugacy classes in } M\}$$

where a tuple (s, T^0) consists of a semisimple element $s \in M$ and a maximal torus T^0 of the fixed-point group $(M^\circ)^{s\circ}$.

Proof. Let (s_1, T_1^0) and (s_2, T_2^0) be two pairs such that $s_1 = gs_2g^{-1}$ for some $g \in M^\circ$. Then both T_1^0 and $gT_2^0g^{-1}$ are maximal tori of the reductive group $(M^\circ)^{s_1\circ}$. So we can choose $h \in (M^\circ)^{s_1\circ}$ such that $hT_1^0h^{-1} = gT_2^0g^{-1}$. Since $hs_1h^{-1} = s_1$, we have $h(s_1, T_1^0)h^{-1} = g(s_2, T_2^0)g^{-1}$. □

2.4.4. Lemma Let $\theta, \theta' \in M$ be semisimple elements such that $\theta^{-1}\theta' \in M^\circ$. Let (B, T) be a θ-stable Borel pair of M°, There exists a conjugate $g\theta'g^{-1}$, $g \in M^\circ$ such that $\theta^{-1}g\theta'g^{-1} \in T$.

Proof. Choose any Borel pair (B^0, T^0) of $(M^\circ)^{\theta'\circ}$. By Theorem 2.4.1, there exists a θ'-stable Borel pair (B', T') such that $(B^0, T^0) = (B', T') \cap M^{\circ\theta'\circ}$. There exists $g \in M^\circ$ such that $g(B', T')g^{-1} = (B, T)$. For ease of notation, we assume $(B', T') = (B, T)$. Now (B, T) is simultaneously θ-stable and θ'-stable. We have $\theta^{-1}\theta' \in N_{M^\circ}(T) \cap N_{M^\circ}(B) = T$. □

2.4.5. Construction of ξ_T Let M be a disconnected reductive group over an algebraically closed field k. Fix a connected component $M^\circ\theta$ of M that that θ is semisimple.

Fix a θ-stable Borel pair (B, T) of M°. Note that θ acts on $X_*(T)$ and $\Omega := N_{M^\circ}(T)/T$. Write $X_*(T)_\theta$ for the θ-coinvariants $X_*(T)/(1-\theta)$, and write Ω^θ for the subgroup of θ-fixed points of Ω. Denote by $X_*(T)_{\theta,\mathrm{tf}}$ the (maximal) torsion-free quotient of the abelian group $X_*(T)_\theta$.

We construct a map

$$\left\{ \begin{array}{l} \text{Semisimple conjugacy classes of M that} \\ \text{lie in the component } M^\circ\theta \end{array} \right\} \xrightarrow{\xi_T} X_*(T)_{\theta,\mathrm{tf}} \otimes k^\times/\Omega^\theta$$

as follows. Let $[\theta']$ be a semisimple conjugacy class of M such that $\theta^{-1}[\theta'] \subset M^\circ$. Define $\xi_T([\theta'])$ to be the equivalence class of $\theta'\theta^{-1} \in T(\bar{\mathbb{F}}_p) = X_*(T) \otimes \bar{\mathbb{F}}_p^\times$.

2.4.6. Proposition If there is a θ-stable pinning $(M^\circ, B, T, \{u_\alpha\}_{\alpha\in R(B,T)})$, then the map ξ_T in Paragraph 2.4.5 is well-defined and is a bijection.

Moreover, each semisimple conjugacy class of M that lies in the component $M^\circ\theta$ admits a representative in $T^{\theta\circ}\theta$.

Proof. Let $[\theta']$ be a semisimple conjugacy class contained in $M^\circ\theta$. By Lemma 2.4.4, there exists a representative θ' such that $t = \theta'\theta^{-1} \in T$. Now $T^{\theta'\circ} = T^{\theta(\theta^{-1}t\theta)\circ} = T^{\theta\circ}$. Write T^0 for $T^{\theta\circ}$. By Theorem 2.4.1, T^0 is a maximal torus of both $M^{\circ\theta\circ}$ and $M^{\circ\theta'\circ}$. By Lemma 2.4.3, if θ_1' is another representative of $[\theta']$ such that $t_1 := \theta_1'\theta^{-1} \in T$, then there exists an element $g \in M^\circ$ such that $g(\theta_1', T^0)g^{-1} = (\theta', T^0)$; in other words, $g \in N_{M^\circ}(T^0)$. By Theorem 2.4.1, $Z_{M^\circ}(T^0) = T$, and thus $N_{M^\circ}(T^0) \subset N_{M^\circ}(T)$.

So far, we have shown ξ_T defines a bijection onto $(X_*(T) \otimes k^\times)/N_{M^\circ}(T^\circ)(k)$, with the caveat that the action of $N_{M^\circ}(T^\circ)(k)$ is θ-twisted, that is,

$$w \cdot t = wt\theta w^{-1}\theta^{-1}, \tag{1}$$

$t \in T$, $w \in N_{M^\circ}(T^\circ)(k)$. Next we show

$$(X_*(T) \otimes k^\times)/T(k) = X_*(T)_\theta \otimes k^\times.$$

Let $s \in T(k)$, and $\theta' \in T(k)\theta$. We have

$$(s\theta's^{-1})\theta^{-1} = s(\theta'\theta^{-1})(\theta s^{-1}\theta^{-1}) = s(\theta s^{-1}\theta^{-1})(\theta'\theta^{-1}).$$

Thus $(X_*(T) \otimes k^\times)/T(k) = \frac{X_*(T)\otimes k^\times}{(1-\theta)X_*(T)\otimes k^\times}$. Since $-\otimes k^\times$ is right-exact, we have $\frac{X_*(T)\otimes k^\times}{(1-\theta)X_*(T)\otimes k^\times} = \frac{X_*(T)}{(1-\theta)X_*(T)} \otimes k^\times$. Write $(-)_{\mathrm{tors}}$ for the torsion part of an abelian group. There exists a short exact sequence

$$0 \to X_*(T)_{\theta,\mathrm{tors}} \to X_*(T)_\theta \to X_*(T)_{\theta,\mathrm{tf}} \to 0$$

where $X_*(T)_{\theta,\mathrm{tf}}$ is the maximal torsion-free quotient of $X_*(T)_\theta$. Since $k^\times$ is a divisible abelian group, we have

$$X_*(T)_\theta \otimes k^\times = X_*(T)_{\theta,\mathrm{tf}} \otimes k^\times.$$

By part (4) of Theorem 2.4.1, we have

$$N_{M^\circ}(T^\circ)/T = (N_{M^\circ}(T)/T)^\theta = N_{M^{\theta\circ}}(T^0)/T^0.$$

Thus ξ_T defines a bijection onto

$$(X_*(T)_\theta \otimes k^\times)/(N_{M^{\theta\circ}}(T^0)/T^0).$$

Moreover, the θ-twisted action (1) becomes untwisted; indeed, for $t \in T$ and $\bar{w} \in N_{M^{\theta\circ}}(T^0)/T^0$, we have $wt\theta w^{-1}\theta^{-1} = wtw^{-1}$, where $w \in N_{M^{\theta\circ}}(T^0)$ is a representative of $\bar{w}$.

Claim $X_*(T^0) \otimes k^\times \to X_*(T)_{\theta,\mathrm{tf}} \otimes k^\times$ is surjective.

Proof. By [Sp98, Proposition 13.2.4], T is an almost direct product T^0T_a such that $T^0 \cap T_a$ is finite. Since $X_*(T)_{\theta,\mathrm{tf}}$ is torsion-free, it defines a quotient torus T_0 of T. The composite $T_a \to T \to T_0$ must be trivial (if otherwise $T_a^{\theta\circ}$ is non-trivial). The composite $T^0 \to T^0T_a/T_a \to T_0$ is thus surjective. □

Translating the claim to a statement about conjugacy classes, we see there exists $s \in T$ such that $s\theta' s^{-1} \in T^0\theta = \theta T^0$. □

2.5. θ-twisted semisimple conjugacy classes

Sometimes it is notationally easier to discuss θ-twisted conjugacy classes than conjugacy classes in a disconnected group. We keep notations and assumptions that are used in Proposition 2.4.6.

In this subsection, M is defined over an algebraically closed field k of characteristic p. Let q be a power of p. Write $\mathrm{Frob}_q : x \mapsto x^q$ be the q-power map.

2.5.1. Definition Two elements $g, g' \in M^\circ$ are said to be θ-conjugate if $g\theta$ and $g'\theta$ are M°-conjugate. Concretely, it means there exists $h \in M^\circ$ such that $g' = hg(\theta h^{-1}\theta^{-1})$. Denote by $[g]_\theta$ the θ-twisted conjugacy class of g.

2.5.2. Definition A set-theoretic map $F : M^\circ \to M^\circ$ is said to be a *θ-twisted Frobenius endomorphism* if

(QF1) $F(T) \subset T$;
(QF2) There exists an automorphism $\varphi \in \mathrm{End}_{\mathbb{Z}}(X_*(T))$ such that $F|_{T^{\theta\circ}} = \varphi \otimes \mathrm{Frob}_q$ under the identification $T = X_*(T) \otimes k^\times$;
(QF3) If $y \in [x]_\theta$, then $F(y) \in [F(x)]_\theta$;
(QF4) $F(\theta^q x\theta^{-q}) = \theta F(x)\theta^{-1}$ for all $x \in M^\circ$.

We say $[g]_\theta$ is F-stable if $F(g') \in [g]_\theta$ for all $g' \in [g]_\theta$.

2.5.3. Proposition If $F : M^\circ \to M^\circ$ is a θ-twisted Frobenius endomorphism, then there exists a bijection

$$\left\{ \begin{array}{l} F\text{-stable } \theta\text{-twisted semisimple conjugacy} \\ \text{classes in } M^\circ \end{array} \right\} \cong (X_*(T)_{\theta,\mathrm{tf}} \otimes k^\times/\Omega^\theta)^{\varphi\otimes\mathrm{Frob}_q}.$$

Proof. By Proposition 2.4.6, the bijection is automatic if the φ-action descends to $X_*(T)_{\theta,\mathrm{tf}}$, which follows immediately from Definition 2.5.2. □

3. Mod p Langlands-Shelstad factorization

Contents

Let G be a connected quasi-split reductive group over F. Fix an F-pinning $(B, T, \{X_\alpha\})$ of G. The pinned group $(G, B, T, \{X_\alpha\})$ has a dual pinned group $(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$ defined over $\mathbb{Z}$, together with an isomorphism of based root data $\Psi_0(\widehat{G}, \widehat{B}, \widehat{T}) \cong \Psi_0(G, B, T)^\vee$.

Let $L \subset F^s$ be a splitting field of G, that is, a Galois subfield of F^s such that the Galois action on $\Psi_0(G_{F^s}, B_{F^s}, T_{F^s})$ factors through $\mathrm{Gal}(L/F)$. The Galois action on the based root datum induces a Galois action on the corresponding pinned reductive group $\mathrm{Gal}(L/F) \to \mathrm{Aut}(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$. Write ${}^L G$ for the semi-direct product $\widehat{G} \rtimes \mathrm{Gal}(L/F)$. See [Zhu21, Section 1.1] and [BG14] for more details.

3.1. Four types of L-groups

In the literature, four different kinds of L-groups are used, and depending on the context, they are not always interchangeable. We list these L-groups as follows:

Galois form	$\widehat{G} \rtimes \mathrm{Gal}(L/F)$
Absolute Galois form	$\widehat{G} \rtimes \mathrm{Gal}_F$
Weil form	$\widehat{G} \rtimes W_F$
Relative Weil form	$\widehat{G} \rtimes W_{L/F}$

3.1.1. Definition Let A be a ring.

A group homomorphism
$$f(A) : \widehat{G}(A) \rtimes \mathrm{Gal}(L/F) \to \widehat{H}(A) \rtimes \mathrm{Gal}(L/F)$$
is said to be an *L-homomorphism* if it sends $g \times \sigma$ to $h \times \sigma$ for all $\sigma \in \mathrm{Gal}(L/F)$.

A group homomorphism
$$f(A) : \widehat{G}(A) \rtimes \mathrm{Gal}_F \to \widehat{H}(A) \rtimes \mathrm{Gal}_F$$
is said to be an *L-homomorphism* if it sends $g \times \sigma$ to $h \times \sigma$ for all $\sigma \in \mathrm{Gal}_F$, and there exists an open subgroup $\Gamma \subset \mathrm{Gal}_F$ such that $1 \times \sigma$ is sent to $1 \times \sigma$ for all $\sigma \in \Gamma$.

A group homomorphism
$$f(A) : \widehat{G}(A) \rtimes W_F \to \widehat{H}(A) \rtimes W_F$$
is said to be an *L-homomorphism* if it sends $g \times \sigma$ to $h \times \sigma$ for all $\sigma \in \mathrm{Gal}_F$, and there exists an open subgroup $\Gamma \subset I_F$ such that $1 \times \sigma$ is sent to $1 \times \sigma$ for all $\sigma \in \Gamma$.

A group homomorphism
$$f(A) : \widehat{G}(A) \rtimes W_{L/F} \to \widehat{H}(A) \rtimes W_{L/F}$$
is said to be an *L-homomorphism* if it sends $g \times \sigma$ to $h \times \sigma$ for all $\sigma \in W_{L/F}$.

3.1.2. Lemma Let A be a finite ring.

(1) If

$$f(A) : \widehat{G}(A) \rtimes \mathrm{Gal}_F \to \widehat{H}(A) \rtimes \mathrm{Gal}_F$$

is an L-homomorphism, then the set $\sigma \in \mathrm{Gal}_F$ such that $f(A)(1 \times \sigma) = 1 \times \sigma$ is an open normal subgroup.

(2) If

$$\widehat{G}(A) \rtimes \mathrm{Gal}_F \to \widehat{H}(A) \rtimes \mathrm{Gal}_F$$

is an L-homomorphism, then it descends to

$$\widehat{G}(A) \rtimes \mathrm{Gal}(L/F) \to \widehat{H}(A) \rtimes \mathrm{Gal}(L/F)$$

for some finite extension L/F.

(3) If

$$\widehat{G}(A) \rtimes \mathrm{Gal}(L/F) \to \widehat{H}(A) \rtimes \mathrm{Gal}(L/F)$$

is an L-homomorphism, then it can be uniquely lifted to an L-homomorphism

$$\widehat{G}(A) \rtimes \mathrm{Gal}_F \to \widehat{H}(A) \rtimes \mathrm{Gal}_F\,.$$

(4) If

$$\widehat{G}(A) \rtimes W_F \to \widehat{H}(A) \rtimes W_F$$

is an L-homomorphism, then it descends to

$$\widehat{G}(A) \rtimes \mathrm{Gal}(L/F) \to \widehat{H}(A) \rtimes \mathrm{Gal}(L/F)$$

for some finite extension L/F.

(5) If

$$\widehat{G}(A) \rtimes \mathrm{Gal}(L/F) \to \widehat{H}(A) \rtimes \mathrm{Gal}(L/F)$$

is an L-homomorphism, then it can be uniquely lifted to an L-homomorphism

$$\widehat{G}(A) \rtimes W_F \to \widehat{H}(A) \rtimes W_F.$$

Proof. (1) Note that $(1 \times \sigma_1)(1 \times \sigma_2) = (1 \times \sigma_1\sigma_2)$.

(2) In a profinite group, an open subgroup is closed of finite index. So part (2) follows from part (1).

(3) Let $\sigma \in \mathrm{Gal}_F$ and write $\bar{\sigma}$ for its image in $\mathrm{Gal}(L/F)$. If $g \times \bar{\sigma} \mapsto h \times \bar{\sigma}$, set $g \times \sigma \mapsto h \times \sigma$. It is easy to check it is a well-defined homomorphism.

(4) Let $\theta \in W_F$ be a Frobenius element, and say $1 \times \theta \mapsto h \times \theta$. Since $\widehat{H}(A)$ is a finite group, there exists an integer n such that $(h \times \theta)^n = 1 \times \theta^n$ acts trivially on $\widehat{H}(A)$. So there exists an open subgroup Γ of W_F of finite index such that $1 \times \sigma \mapsto 1 \times \sigma$ for all $\sigma \in \Gamma$ and that Γ acts trivially on $\widehat{H}(A)$.

(5) It is similar to part (3). $\square$

By Lemma 3.1.2, the various forms of L-groups make little difference when working with finite coefficients.

3.1.3. Definition Let A be a finite ring equipped with discrete topology, and let G and H be two reductive groups over F. Let L be a sufficiently large field extension of F that splits both G and H.

An L-morphism $f : {}^L G_A \to {}^L H_A$ is an algebraic group morphism over A such that for all A-algebras B with finitely many elements, $f(B)$ is an L-homomorphism.

Write ${}^L\{*\}$ for the L-group of the trivial group. An *L-parameter for G with A-coefficients* is an L-morphism ${}^L\{*\} \to {}^L G_A$, defined up to $\widehat{G}(A)$-conjugacy. Equivalently, since ${}^L\{*\}$ is a constant group scheme, an L-parameter can also be defined as an L-homomorphism $\mathrm{Gal}_F \to {}^L G(A)$, defined up to $\widehat{G}(A)$-conjugacy.

Let R be a profinite ring. A *profinite L-parameter for G with R-coefficients* is a compatible system of L-parameters $\mathrm{Gal}_F \to {}^L G(A)$, where A is a finite quotient of R.

From now on, assume G is tamely ramified (that is, the splitting field L can be chosen as a tame extension of F).

3.2. Quasi-semisimplicity of semisimple mod p L-parameters

3.2.1. Standard parabolic subgroups and Levi subgroups of Langlands dual groups Write $\widehat{\Delta}$ for $\Delta(\widehat{B}, \widehat{T})$. Any $\mathrm{Gal}(L/F)$-stable subset $S \subset \widehat{\Delta}$ corresponds to a pinned subgroup $(M_S, M_S \cap \widehat{B}, \widehat{T}, \{Y_\alpha\}|_S) \subset (\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$ which is stable under the $\mathrm{Gal}(L/F)$-action. Write P_S for a parabolic subgroup of $\widehat{G}$ having M_S as a Levi subgroup. We call $M_S \rtimes \mathrm{Gal}(L/F)$ a *standard Levi subgroup* of ${}^L G$, and we call $P_S \rtimes \mathrm{Gal}(L/F)$ a *standard parabolic subgroup* of ${}^L G$.

3.2.2. Lemma All big pseudo-parabolics of ${}^L G$ are conjugate to a standard parabolic.

Proof. Let P be a big pseudo-parabolic of ${}^L G$. There exists a Borel subgroup B of P°. Let B_{std} be the standard parabolic whose neutral component is a Borel. Then there exists an element $g \in \widehat{G}$ such that $gBg^{-1} = B^\circ_{\mathrm{std}}$. After replacing B_{std} by $g^{-1}B_{\mathrm{std}}g$, we can assume $B \subset B_{\mathrm{std}}$. For each $\bar{\gamma} \in \mathrm{Gal}(L/F)$, let $\gamma \in P$ be a lift of γ in P and let $\gamma_{\mathrm{std}} \in B_{\mathrm{std}}$ be the "standard" lift of γ in B_{std} (note that $B_{\mathrm{std}} = B \rtimes \mathrm{Gal}(L/F)$ admits a distinguished copy of $\mathrm{Gal}(L/F)$). There exists $h \in P$ such that $\gamma B \gamma^{-1} = hBh^{-1}$. We have $h^{-1}\gamma\gamma_{\mathrm{std}}^{-1}(B)\gamma_{\mathrm{std}}\gamma^{-1}h = h^{-1}\gamma(B)\gamma^{-1}h = B$. Since B is self-normalizing in $\widehat{G}$, we have $h^{-1}\gamma\gamma_{\mathrm{std}}^{-1} \in B$. Thus $\gamma_{\mathrm{std}} \in P$.

The same argument applies to all $\bar{\gamma} \in \mathrm{Gal}(L/F)$. So $B_{\mathrm{std}} \subset P$. Now it is clear that all positive simple roots of P with respect to the standard $(\widehat{B}, \widehat{T})$-pair are permuted by the $\mathrm{Gal}(L/F)$-action, and P is a standard parabolic. $\square$

3.2.3. Definition For a ring k, a *parabolic subgroup* of ${}^L G_k$ is a subgroup ${}^L P$ conjugate to a standard parabolic of ${}^L G_k$. Equivalently, a parabolic of ${}^L G_k$ is a big pseudo-parabolic.

A *Levi subgroup* of a parabolic ${}^L P$ is a subgroup which is conjugate to a standard Levi of ${}^L G$.

3.2.4. Remark We will avoid using the term "parabolic subgroup" because it is not consistent with the usual definition that P is parabolic if G/P is a projective variety.

Instead, we prefer the use "big pseudo-parabolic" to avoid confusions.

We also remark that the lemma above is true only because (1) the group ${}^L G$ is itself a semi-direct product, and (2) (for connected groups) parabolic subgroups are self-normalizing. For a general (disconnected) reductive subgroup H of ${}^L G$, we should not expect its big pseudo-parabolics to be a split extension of its neutral component, or conjugate to a standard parabolic in general.

3.2.5. Definition Let k be a field. Let $\rho : \mathrm{Gal}_F \to {}^L G(\bar{k})$ be an L-parameter.

- ρ is said to be *pseudo-irreducible* if the image of ρ is pseudo-irreducible (Definition 2.2.3);
- ρ is said to be *pseudo-semisimple* or *semisimple* if the image of ρ is pseudo-completely reducible;
- ρ is said to be *quasi-semisimple* if there exists a maximal torus T of $\widehat{G}$ such that
  - $\rho(I_L) \subset T(k)$, and
  - $\mathrm{Im}\,\rho \subset N_{{}^L G}(T)(k)$.

  The term "quasi-semisimple" appears in [St68, Section 9], which is loosely related to our situation.

3.2.6. Lemma If $\rho : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ is a semisimple L-parameter, then ρ is tamely ramified.

Proof. Let $P_F \subset \mathrm{Gal}_F$ be the wild inertia. The image $\rho(P_F) \subset \widehat{G}(\bar{\mathbb{F}}_p)$ is a p-group, and thus consists of unipotent elements. By [BT71, Corollaire 3.9], there exists a parabolic subgroup P of $\widehat{G}_{\bar{\mathbb{F}}_p}$ with unipotent radical $R_u(P)$ such that

- $\rho(P_K) \subset R_u(P)(\bar{\mathbb{F}}_p)$,
- $N_{\widehat{G}}(\rho(P_K)) \subset P(\bar{\mathbb{F}}_p)$, and
- all automorphism of $\widehat{G}_{\bar{\mathbb{F}}_p}$ which fix $\rho(P_K)$ also fix P.

Here $N_{\widehat{G}}(\rho(P_K))$ is the normalizer of $\rho(P_K)$ in $\widehat{G}$. Since P_K is a normal subgroup of Gal_L, $\rho(\mathrm{Gal}_L) \subset N_{\widehat{G}}(\rho(P_K)) \subset P(\bar{\mathbb{F}}_p)$. Since P_K is a normal subgroup of Gal_F and all automorphism of $\widehat{G}_{\bar{\mathbb{F}}_p}$ which fix $\rho(P_K)$ also fix P, the subset $\Gamma := P(\bar{\mathbb{F}}_p)\rho(\mathrm{Gal}_F) \subset {}^L G(\bar{\mathbb{F}}_p)$ is a subgroup of ${}^L G(\bar{\mathbb{F}}_p)$. We have $\Gamma \cap \widehat{G} = P(\bar{\mathbb{F}}_p)N_{\widehat{G}}(\rho(P_K)) = P(\bar{\mathbb{F}}_p)$. By Lemma 2.2.6, Γ is a big pseudo-parabolic of ${}^L G_{\bar{\mathbb{F}}_p}$. Since ρ is semisimple, ρ factors through a pseudo-Levi M of Γ. So $\rho(P_K) \subset R_u(P)(\bar{\mathbb{F}}_p) \cap M = \{1\}$. □

3.2.7. Theorem Semisimple L-parameters $\rho : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ are quasi-semisimple.

Proof. By Lemma 3.2.6, ρ is tamely ramified. Write $\mathfrak{F}$ for a Frobenius element of Gal_F and let $\mathfrak{T}$ be an element of Gal_F whose image in the tame quotient is a topological generator. Write $\sigma := \rho(\mathfrak{F})$ and $\tau := \rho(\mathfrak{T})$. We have $\sigma\tau\sigma^{-1} = \tau^q$. Since τ has prime-to-p order, τ acts on $\widehat{G}$ semisimply. The theorem follows from Corollary 2.3.7. □

3.2.8. Definition Let ρ be a tamely ramified L-parameter $\mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$. Let ${}^L P$ be minimal among all big pseudo-parabolic of ${}^L G$ through which ρ factors. Let $\pi : {}^L P \to {}^L M$ be the quotient by the unipotent radical U map, and let $\iota : {}^L M \hookrightarrow {}^L P$ be a splitting of the semi-direct product $1 \to U \to {}^L P \to {}^L M \to 1$. We say $\iota \circ \pi \circ \rho$ is a *pseudo-semisimplification* of ρ, and denote it by ρ^{ss}.

3.2.9. Lemma For any tamely ramified L-parameter $\rho : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$, $\rho|_{I_F}$ is conjugate to $\rho^{\mathrm{ss}}|_{I_F}$.

Proof. Let $\theta \in I_F$ be a topological generator of the tame quotient of the inertia. Let ${}^L P$ be minimal among all big pseudo-parabolic of ${}^L G_{\bar{\mathbb{F}}_p}$ through which ρ factors. Since $\rho(\theta)$ is semisimple, it fixes a Borel pair (B_P, T_P) of ${}^L P$ ([St68, Theorem 7.5]). $(\pi(B_P), \pi(T_P))$ is a Borel pair of ${}^L M$ fixed by $\pi(\rho(\theta))$, and thus $(\iota\pi(B_P), \iota\pi(T_P))$ is a Borel pair of $\iota({}^L M)$ fixed by $\rho^{\mathrm{ss}}(\theta)$. Furthermore, $(\iota\pi(B_P)U, \iota\pi(T_P))$ is a Borel pair of ${}^L P$. There exists $g \in U$ such that $g(B_P, T_P)g^{-1} = (\iota\pi(B_P)U, \iota\pi(T_P))$. By replacing ι by a conjugate, we may assume $g = 1$. Write t for $\rho(\theta)\rho^{\mathrm{ss}}(\theta)^{-1} \in ({}^L P)^\circ$. We have $t \in N_{\widehat{G}}(B_P) \cap N_{\widehat{G}}(T_P) = T_P$. By the construction of ρ^{ss}, we have $t \in U$. So $t \in T_P \cap U = \{1\}$. □

3.3. Maximally unramified tori

3.3.1. Convention All homomorphisms of algebraic F-groups $f : G \to H$ are meant to be defined over F^s, unless it is said explicitly that it is defined over F. For ease of notation, we write G for $G(F^s)$ if it is clear from the context that we are treating it as a set.

Recall that we fixed an F-pinning $(B, T, \{X_\alpha\})$ of G. Write Ω for the Weyl group $N_G(T)/T := N_{G(F^s)}(T(F^s))/T(F^s)$, which is canonically identified with $N_{\widehat{G}}(\widehat{T})/\widehat{T}$.

Following [Kal19a, Section 5.1], it is conceptually easier to think about framed maximal tori (see Definition A.0.2 and A.0.4) instead of maximal tori.

Let $j : S \to G$ be a framed maximal tori. By replacing j by a conjugate gjg^{-1}, $g \in G(F^s)$, we may assume $j(S) \subset T$. Thus there is a canonical bijection

$$\left\{ \begin{array}{l} G(F^s)\text{-conjugacy classes of framed max-} \\ \text{imal tori of } G \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \Omega\text{-conjugacy classes of pairs } (S, j) \text{ where} \\ S \text{ is a } F\text{-torus and } j : S_{F^s} \to T_{F^s} \text{ is an} \\ \text{isomorphism of tori} \end{array} \right\}$$

Similarly, there is a bijection on the dual side

$$\left\{ \begin{array}{l} \widehat{G}\text{-conjugacy classes of framed maximal} \\ \text{tori of } {}^LG_{\bar{\mathbb{F}}_p} \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \Omega\text{-conjugacy classes of pairs } (S, \widehat{\jmath}) \text{ where} \\ S \text{ is a } F\text{-torus and } \widehat{\jmath} : \widehat{S} \to \widehat{T} \text{ is an iso-} \\ \text{morphism of tori} \end{array} \right\}$$

By sending $(S, j) \mapsto (S, (\widehat{j})^{-1})$ where $\widehat{j} : \widehat{T} \to \widehat{S}$ is the functorial dual torus map, we get a duality between the geometric conjugacy classes of maximal tori of G and that of ${}^LG_{\bar{\mathbb{F}}_p}$.

Recall the following fact ([Kal19a, Fact 3.4.1]):

3.3.2. Fact Let $S \subset G$ be a maximal torus and let $S_s \subset S$ be the maximal unramified torus (both embeddings are defined over F). The following statements are equivalent.

- S_s is of maximal dimension among the unramified subtori of G.
- S_s is not properly contained in an unramified subtorus of G.
- S is the centralizer of S_s in G.
- The action of I_F on $R(S, G)$ preserves a set of positive roots.

3.3.3. Definition Let $j : S \to G$ be a framed maximal torus. The inertia I_F acts on $X^*(S)$ by $\theta \cdot \alpha := \theta \circ \alpha \circ \theta^{-1}$, and thus acts on $X^*(j(S))$ by transport ($\theta \cdot \alpha := (\theta \cdot (\alpha \circ j)) \circ j^{-1}$). Note that in general the I_F-action on $X^*(j(S))$ does not need to preserve the set of absolute roots $R(j(S), G) \subset X^*(j(S))$.

We say (S, j) is *maximally unramified* if I_F preserves $R(j(S), G)$ and a set of positive roots thereof.

3.3.4. Lemma Let $S \subset G$ be a maximal F-torus. Then the tautological embedding $j : S \hookrightarrow G$ defines a maximally unramified framed maximal torus if and only if S satisfies the equivalent statements in Fact 3.3.2.

Proof. Since j is the tautological embedding of an F-torus, I_F preserves $R(j(S), G)$. The Lemma now follows from the last bullet point of Fact 3.3.2. □

3.3.5. Lemma (S, j) is a maximally unramified framed torus of G if and only if a geometric conjugate of (S, j) is maximally unramified.

Proof. Write $\mathrm{Int}(g)$ for $x \mapsto gxg^{-1}$, $g \in G(F^s)$. We have

$$R(\mathrm{Int}(g) \circ j(S), G) = \{\alpha \circ \mathrm{Int}(g)^{-1} | \alpha \in R(j(S), G)\}.$$

So it is clear I_F preserves $R(\mathrm{Int}(g)\circ j(S), G)$ if and only if it preserves $R(j(S), G)$. If I_F preserves a set of positive roots $R^+ \subset R(j(S), G)$ than $\mathrm{Int}(g)^{-1}(R^+)$ is a set of positive roots of $R(\mathrm{Int}(g)\circ j(S), G)$ that I_F preserves. $\square$

3.3.6. Definition Let $(S, \hat{\jmath})$, $\hat{\jmath} : \widehat{S} \to \widehat{G}$ be a framed maximal torus of ${}^L G$. The I_F-action on $X^*(\widehat{S})$ transports to $X^*(\hat{\jmath}(\widehat{S}))$. We say $(S, \hat{\jmath})$ is maximally unramified if I_F preserves $R(\hat{\jmath}(\widehat{S}), \widehat{G})$ and a set of positive roots thereof.

3.3.7. Lemma Let (S, j) be a framed maximal torus of G whose geometric conjugacy class is $\mathrm{Gal}(F^s/F)$-stable. Then (S, j) is maximally unramified if and only if its dual $(S, \hat{\jmath})$, $\hat{\jmath} : \widehat{S} \to \widehat{G}$ is a maximally unramified framed maximal torus of ${}^L G$.

Proof. By Theorem A.0.3, we can assume $S \subset G$ is an F-subtorus. Let $R^+ \subset R(S, G)$ be a set of positive roots preserved by I_F. R^+ defines a Borel $B \supset S_{F^s}$ of G_{F^s}. Since I_F permutes the root groups U_α ($\alpha \in R^+$) of B, B is I_F-stable. As a consequence, I_F also permutes the positive coroots of B, which translates to the maximal unramifiedness of $(S, \hat{\jmath})$. $\square$

We summarize the duality in the following proposition.

3.3.8. Proposition There is a canonical bijection:

$$\left\{ \begin{array}{l} \mathrm{Gal}_F\text{-stable } G(F^s)\text{-conjugacy classes of} \\ \text{maximally unramified framed maximal} \\ \text{torus of } G \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \mathrm{Gal}_F\text{-stable conjugacy classes of maxi-} \\ \text{mally unramified framed maximal torus} \\ \text{of } {}^L G \end{array} \right\}$$

Recall that a *twisted Levi* of G is a subgroup $M \subset G$ defined over F which becomes a Levi subgroup after bash change to F^s. We have the following fact.

3.3.9. Langlands-Shelstad L-embedding ([Kal19a, Lemma 5.2.6]) Let $M \subset G$ be a tame twisted Levi, and let $\widehat{M}_{\mathbb{C}} \to \widehat{G}_{\mathbb{C}}$ be the natural inclusion. Write ${}^{\mathfrak{L}}M$ and ${}^{\mathfrak{L}}G$ for the Weil form of the L-group. There exists an extension of $\widehat{M} \to \widehat{G}$ to a tame L-embedding ${}^{\mathfrak{L}}M_{\mathbb{C}} \to {}^{\mathfrak{L}}G_{\mathbb{C}}$.

The conjugacy class of the tame L-embedding ${}^{\mathfrak{L}}M_{\mathbb{C}} \to {}^{\mathfrak{L}}G_{\mathbb{C}}$ depends on the choice of the so-called χ-*data*. We remark that it is important to use the Weil form. An L-embedding ${}^L M_{\mathbb{C}} \to {}^L G_{\mathbb{C}}$ of the absolute Galois form does not exist in general (see [Kal21]).

When $M \subset G$ be a maximally unramified torus (Definition 3.3.3), there is a canonical choice of χ-data (the unramified χ-data). We establish the integral version in the following theorem.

3.3.10. Theorem Let $S \subset G$ be a maximal F-torus such that the tautological embedding $S \hookrightarrow$ defines a maximally unramified framed maximal torus.

For each choice of Borel $B_S \supset S$ of G defined over F^s, there exists an element $g \in G(F^s)$ such that $(B_S, S) = g(B, T)g^{-1}$ (recall that we fixed a pinning $(B, T, \{X_\alpha\})$ of G). The conjugation-by-g map $\mathrm{Int}(g) : T \to S$ (which is only defined over F^s) has a dual map $\widehat{\mathrm{Int}(g)} : \widehat{S} \to \widehat{T}$.

Let A be a finite ring. Feeding the unramified χ-data into the recipe in [LS87, Section 2.6], we get an L-embedding ${}^L j : {}^L S_A \to {}^L G_A$ extending the natural embedding $\widehat{S} \xrightarrow{\widehat{\mathrm{Int}(g)}} \widehat{T} \subset \widehat{G}$.

Moreover, ${}^L j$ is uniquely determined up to $\widehat{T}$-conjugacy by the choice of

- the Gal_F-pinning $(\widehat{B}, \widehat{T}, \{Y_\alpha\})$ of $\widehat{G}$, and
- the Borel B_S.

Proof. By Lemma 3.1.2, it suffices to produce an L-embedding for the Weil form of L-groups. We refer the reader to [Kal19a, Section 2.2] for the notion of ramified/unramified symmetric/asymmetric roots, and the definition of χ-data. Since I_F stabilizes a set of positive roots (Definition 3.3.3), there is no symmetric ramified root (the existence of symmetric ramified roots implies there exists a ramified quadratic extension E/F such that the $\mathrm{Gal}(E/F)$-orbit of a positive root α is $\pm\alpha$). Consequently, we can choose the unramified χ-data $\{\chi_\alpha\}$ such that $\chi_\alpha = 1$ if α is asymmetric and χ_α is the unramified quadratic character if otherwise. In particular, all these characters χ_α are valued in $\{\pm 1\}$. We remark that unramified χ-data are minimally ramified in the sense of [Kal19a, Definition 4.6.1].

So it suffices to show the Langlands-Shelstad L-embedding in [LS87, Section 2] for unramified χ-data is defined over A. The formula in [LS87, Section 2.6] for the L-embedding reads

$$\xi(w) = r_p(w)n(\omega_T(\sigma)) \times w$$

(see loc. cit. for the notations). Note that $n(\omega_T(\sigma))$ is a product of the image of root vectors under the exponential map (see [LS87, Section 2.1]) and the exponential map is well-defined for split group schemes over an arbitrary base scheme [Crd11, Theorem 4.1.4]. The map $r_p(w)$ has explicit formula in [LS87, Section 2.5] and it is valued in $\{\pm 1\} \otimes X^*(T)$ because all the characters χ_α are valued in $\{\pm 1\}$. Since $\{\pm 1\} \subset A^\times$ for an arbitrary finite ring A, we have $\{\pm 1\} \otimes X^*(T) \subset A^\times \otimes X^*(T) = \widehat{T}(A)$. In particular, ξ is defined over A. The "moreover" part is the remark right above [LS87, Paragraph 2.6.1]. $\square$

For lack of a better terminology, we call L-embeddings ${}^LS \to {}^LG$ that appear in the theorem above *canonical L-embeddings*.

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ. Taking the inverse limit of ${}^LS(\mathcal{O}/\varpi^n) \to {}^LG(\mathcal{O}/\varpi^n)$, we get an L-embedding ${}^LS(\mathcal{O}) \to {}^LG(\mathcal{O})$. Taking the union of ${}^LS(\mathcal{O}) \to {}^LG(\mathcal{O})$ as E varies, we get an L-embedding ${}^LS(\bar{\mathbb{Z}}_p) \to {}^LG(\bar{\mathbb{Z}}_p)$. As we have remarked before, we don't think an L-embedding ${}^LS(\bar{\mathbb{Q}}_p) \to {}^LG(\bar{\mathbb{Q}}_p)$ exists in general ([Kal21]).

3.4. Langlands-Shelstad factorization of L-parameters

3.4.1. Theorem For each semisimple L-parameter $\rho : \mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$, there exists a maximally unramified tame torus S of G defined over F and a canonical L-homomorphism ${}^Lj : {}^LS_{\bar{\mathbb{F}}_p} \to {}^LG_{\bar{\mathbb{F}}_p}$ such that ρ factors through ${}^Lj$.

As a byproduct of the construction, we can ensure the ramification index of S is equal to that of G and that there exists a $\rho(I_F)$-stable Borel $\widehat{B} \subset \widehat{G}$ that contains $\widehat{S}$.

Proof. Write $\sigma \in \mathrm{Gal}_F$ for a Frobenius element, and write $\tau \in \mathrm{Gal}_F$ for an element generating the tame quotient of the inertia.

By Corollary 2.3.7 and Theorem 3.2.7, there exists a maximal torus $\widehat{S} \subset \widehat{G}$ which is stable under $\rho(\sigma)$ and $\rho(\tau)$, and a $\rho(\tau)$-stable Borel $\widehat{B}$ of $\widehat{G}$ containing $\widehat{S}$. Thus the torus $\widehat{S}$, equipped with Galois action $\gamma \cdot x := \rho(\gamma)x\rho(\gamma)^{-1}$ ($x \in S$, $\gamma \in \mathrm{Gal}_F$), together with the tautological embedding $\widehat{S} \to \widehat{G}$ is maximally unramified in the sense of Definition 3.3.6. By Proposition 3.3.8 and Theorem A.0.3, S exists. By Theorem 3.3.10, ${}^Lj$ exists. Write $\varphi : \mathrm{Gal}_F \to {}^LS(\bar{\mathbb{F}}_p)$ the map $\gamma \mapsto 1 \rtimes \gamma$. Write $\rho' := {}^Lj \circ \varphi$. Note that for each $\gamma \in \mathrm{Gal}_F$, $\rho(\gamma)\rho'(\gamma)^{-1}$ acts trivially on $\widehat{S}$, and thus $\rho(\gamma)\rho'(\gamma)^{-1} \in Z_{\widehat{G}}(\widehat{S}) = \widehat{S}$. So $\mathrm{Im}\,\rho \subset {}^Lj({}^LS(\bar{\mathbb{F}}_p))$. Since ${}^Lj$ is an embedding, we have a factorization as desired.

Write e for the ramification index of G, we have $\rho(\tau)^e(1 \rtimes \tau^{-e}) \in \widehat{G} \cap N_{{}^L G_{\bar{\mathbb{F}}_p}}(\widehat{B}) \cap N_{{}^L G_{\bar{\mathbb{F}}_p}}(\widehat{S}) = \widehat{S}$, which implies the ramification index of S is at most e. $\square$

4. The Deligne-Lusztig map

Contents

The uniqueness of canonical L-embeddings (see Theorem 3.3.10) allows us to attach characters of F-tori of G to semisimple L-parameters for G.

We start with the following basic lemma, which claims that homomorphisms of tori which respect F-points are necessarily defined over F.

4.0.1. Lemma Let T and S be two F-tori. Let $f : T_{F^s} \to S_{F^s}$ be an isomorphism of F^s-tori such that $f(T(F)) = S(F)$. Then f is an F-isomorphism.

Proof. Let L/F be a finite extension splitting both T and S. We may assume f is an L-isomorphism between T_L and S_L. Write

$$\begin{aligned} j_T &: T \to \operatorname{Res}_{L/F} T_L \\ j_S &: S \to \operatorname{Res}_{L/F} S_L \end{aligned}$$

for the adjunction maps relating Weil restriction and base change. On the level of functor of points, the map

$$j_T(F) : T(F) \to (\operatorname{Res}_{L/F} T_L)(F) = T(L \otimes_F F) = T(L)$$

is the map induced by the inclusion $F \subset L$ ([CGP15, Proposition A.5.7]). By assumption, we have $\operatorname{Res}_{L/F} f(j_T(T(F))) = j_S(S(F))$. By [CGP15, Proposition A.5.7], j_T is a closed immersion defined over F. It is clear that $\operatorname{Res}_{L/F} f$ is an F-isomorphism. Write $j(T)$ for the subtorus $\operatorname{Res}_{L/F} f(j_T(T)) \subset \operatorname{Res}_{L/F} S_L$, and write $j(S)$ for the subtorus $j_S(S) \subset \operatorname{Res}_{L/F} S_L$. Note that both $j(T)$ and $j(S)$ are F-subtorus of $\operatorname{Res}_{L/F} S_L$. We have shown $j(T)(F) = j(S)(F)$. By [Sp98, Corollary 13.3.10], since F is an infinite field, $j(T)(F)$ is Zariski dense in $j(T)$; and $j(S)(F)$ is Zariski dense in $j(S)$. Thus $j(S) = j(T)$. Now $T \cong j(T) = j(S) \cong S$ as an F-torus. $\square$

4.1. Stable conjugacy Fix an F-pinning $(B, T, \{X_\alpha\})$ of G and its dual pinning $(\widehat{B}, \widehat{T}, \{Y_\alpha\})$ once for all.

4.1.1. Definition A *Deligne-Lusztig datum* is a pair (S,χ) consisting of a maximally unramified maximal F-torus $S \subset G$ and a character $\chi : S(F) \to \bar{\mathbb{F}}_p^\times$.

Write $\mathrm{Int}(g)$ for the conjugation by g map $x \mapsto gxg^{-1}$.

Two Deligne-Lusztig data (DL data for short) (S_1,χ_1) and (S_2,χ_2) are said to be stably conjugate if there exists an element $g \in G(F^s)$ such that $S_2(F) = gS_1(F)g^{-1}$ and $\chi_2 = \mathrm{Int}(g)_*\chi_1$. By Lemma 4.0.1, $\mathrm{Int}(g)$ is an F-isomorphism.

A *based Deligne-Lusztig datum* is a tuple (S,χ,B_S) where (S,χ) is a Deligne-Lusztig datum and $B_S \subset G_{F^s}$ is a Borel defined over F^s containing S.

4.1.2. The Deligne-Lusztig map By Theorem 3.3.10, given a Borel pair (S,B_S), there is a canonical L-embedding ${}^Lj : {}^LS_{\bar{\mathbb{F}}_p} \to {}^LG_{\bar{\mathbb{F}}_p}$, unique up to $\widehat{T}$-conjugacy. Given a DL datum (S,χ), by the Local Langlands Correspondence for tori, we can attach to it an L-homomorphism $\rho_\chi : \mathrm{Gal}_F \to {}^LS(\bar{\mathbb{F}}_p)$, well-defined up to $\widehat{S}$-conjugacy. Define the *Deligne-Lusztig map*

$$\mathrm{DL} : (S,\chi,B_S) \mapsto {}^Lj \circ \rho_\chi,$$

which is well-defined up to $\widehat{T}$-conjugacy.

4.1.3. Lemma Let (S,χ) be a DL datum and let (S,χ,B_S) and (S,χ,B'_S) be two enhancements of (S,χ). Then $\mathrm{DL}(S,\chi,B_S)$ and $\mathrm{DL}(S,\chi,B'_S)$ are $\widehat{G}$-conjugate to each other.

Proof. By [LS87, Lemma 2.6.A], the $\widehat{G}$-conjugacy class of ${}^Lj : {}^LS_{\bar{\mathbb{F}}_p} \to {}^LG_{\bar{\mathbb{F}}_p}$ does not depend on the choice of B_S. See the proof of Theorem 3.3.10 for characteristic p issues. □

As a consequence of the lemma above, for each DL datum (S,χ), $\mathrm{DL}(S,\chi) := \mathrm{DL}(S,\chi,B_S)$ is well-defined up to $\widehat{G}$-conjugacy.

4.1.4. Lemma Let (S,χ) and (S',χ') be two stably conjugate DL data. Then $\mathrm{DL}(S,\chi)$ and $\mathrm{DL}(S',\chi')$ are $\widehat{G}$-conjugate.

Proof. Let (S,χ,B_S) be an enhancement of (S,χ). Let $g \in G(F^s)$ be an element such that $gS(F)g^{-1} = S'(F)$ and $\chi' = \mathrm{Int}(g)_*\chi$. Set $B'_S := gB_Sg^{-1}$. By Lemma 4.0.1, $\mathrm{Int}(g) : S \to S'$ is an F-isomorphism and thus induces an isomorphism ${}^L\mathrm{Int}(g) : {}^LS' \xrightarrow{\cong} {}^LS$ of L-groups.

By the functoriality of the LLC for tori, $\rho_\chi = {}^L\mathrm{Int}(g) \circ \rho_{\chi'}$. Thus $\mathrm{DL}(S,\chi,B_S) = \mathrm{DL}(S',\chi',B'_S)$. The lemma now follows from the definition of $\mathrm{DL}(S,\chi)$. □

4.1.5. Proposition The map DL induces a surjective map from the set of stable conjugacy classes of DL data (S,χ) to the set of semisimple L-parameters $\mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$.

Proof. The well-definedness follows from Lemma 4.1.4 and the surjectivity follows from Theorem 3.4.1. □

4.1.6. Failure of injectivity

The map DL in Proposition 4.1.5 is not injective in general.

Consider the spherical L-parameter $\rho : \mathrm{Gal}_F \to \mathrm{GL}_2$ for GL_2 which is trivial on the inertial I_F and sends a Frobenius element to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. On the one hand, ρ factors through the L-group of an elliptic maximal torus S_e of GL_2. On the other hand, since the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is diagonalizable, ρ also factors through the L-group of a split maximal torus S_s of GL_2. Therefore, two DL data (S_e, χ_e) and (S_s, χ_s) are both sent to ρ under DL. However, since S_e and S_s are not F-isomorphic, by Lemma 4.0.1, (S_e, χ_e) and (S_s, χ_s) are not stably conjugate.

Denote by $\breve{F}$ the strict henselization of F. Note that both S_s and S_e are maximally unramified maximal torus of GL_2 (indeed, they are both unramified), and they both become split after base change to $\breve{F}$. Since any two split maximal tori are rationally conjugate to each other, there exists an element $g \in G(\breve{F})$ such that $gS_s(\breve{F})g^{-1} = S_e(\breve{F})$. This observation suggests that the inertial version of DL has a chance to be a bijection.

4.2. Characters of tori over finite fields

4.2.1. Lemma Let X be a finite free abelian group equipped with a finite order automorphism π. There exists isomorphisms

$$\frac{X}{(q-\pi)X} \cong (X \otimes \mathbb{Q}_{p'}/\mathbb{Z})^{q\pi^{-1}}$$
$$\frac{X^\vee}{(q-\pi)X^\vee} \cong \mathrm{Hom}((X \otimes \mathbb{Q}_{p'}/\mathbb{Z})^{q\pi^{-1}}, \mathbb{Q}_{p'}/\mathbb{Z})$$

where $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$.

Proof. It is [Ca93, Proposition 3.2.2] and [Ca93, Proposition 3.2.3]. $\square$

Indeed, for each X as in the lemma above, there exists a (unique up to isomorphism) torus $\underline{T}$ defined over $\mathbb{F}_q$ such that $X = X_*(\underline{T})$ (Proposition A.0.1), and there is a isomorphism $(X \otimes \mathbb{Q}_{p'}/\mathbb{Z})^{q\pi^{-1}} \cong (X \otimes \bar{\mathbb{F}}_p^\times)^{\pi^{-1} \otimes \mathrm{Frob}_q} = \underline{T}(\mathbb{F}_q)$. By the lemma above, there are short exact sequences

$$0 \to X_*(\underline{T}) \xrightarrow{q-\pi} X_*(\underline{T}) \xrightarrow{\Xi} \underline{T}(\mathbb{F}_q) \to 0, \text{ and} \tag{2}$$

$$0 \to X^*(\underline{T}) \xrightarrow{q-\pi} X^*(\underline{T}) \xrightarrow{\Xi} \mathrm{Hom}(\underline{T}(\mathbb{F}_q), \bar{\mathbb{F}}_p^\times) \to 0. \tag{3}$$

The short exact sequence (2) enables us to "forget the $\mathbb{F}_q$-structure" on $\underline{T}$ when working with characters of $\underline{T}(\mathbb{F}_q)$. Given a continuous homomorphism

$$\chi : \underline{T}(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times,$$

the composition $\chi \circ \Xi$ is a character of the free abelian group $X_*(\underline{T})$, which is independent of the $\mathbb{F}_q$-structure on $\underline{T}$. Moreover the composite $\chi \circ \Xi$ uniquely determines the original character χ, once the $\mathbb{F}_q$-structure on $\underline{T}$ is specified.

We write $\underline{T}_\pi(\mathbb{F}_q) = \underline{T}(\mathbb{F}_q)$ to emphasize the $\mathbb{F}_q$-points are taken with respect to π.

4.2.2. Example The 1-dimensional torus $\underline{\mathbb{G}_m}$ over $\bar{\mathbb{F}}_p$ admits two $\mathbb{F}_q$-structures π and π' that splits over $\mathbb{F}_{q^2}$: π corresponds to the split 1-dim $\mathbb{F}_q$-torus and π' corresponds to the (unique up to isomorphism) nonsplit 1-dim $\mathbb{F}_q$-torus. Let

$$\chi : \underline{\mathbb{G}_m}_\pi(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times$$

and

$$\chi' : \underline{\mathbb{G}_m}_{\pi'}(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times$$

be trivial characters $x \mapsto 1$. Even though $\underline{\mathbb{G}_m}_\pi(\mathbb{F}_q)$ and $\underline{\mathbb{G}_m}_{\pi'}(\mathbb{F}_q)$ are distinct subsets of $\underline{\mathbb{G}_m}(\bar{\mathbb{F}}_p)$, we have $\chi \circ \Xi = \chi' \circ \Xi' = 1$.

4.2.3. Definition Let $\underline{T}$ be a torus over $\bar{\mathbb{F}}_p$ equipped with two (possibly different) $\mathbb{F}_q$-structures π and π'. A character $\chi : \underline{T}_\pi(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times$ and a character $\chi' : \underline{T}_{\pi'}(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times$ are said to be equivalent if they define the same character of $X_*(\underline{T})$. We write $\chi \cong \chi'$ if they are equivalent.

4.2.4. Definition Let $\underline{S}_1$, $\underline{S}_2$ be two tori defined over a field $\mathbb{F}_q$. Let $\underline{\chi}_1 : \underline{S}_1(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times$ and $\underline{\chi}_2 : \underline{S}_2(\mathbb{F}_q) \to \bar{\mathbb{F}}_p^\times$ be characters.

A $\bar{\mathbb{F}}_p$-isomorphism $f : (\underline{S}_1)_{\bar{\mathbb{F}}_p} \to (\underline{S}_2)_{\bar{\mathbb{F}}_p}$ is said to define an equivalence of $(\underline{S}_1, \underline{\chi}_1)$ and $(\underline{S}_2, \underline{\chi}_2)$ if $f_*(\chi_1) \cong \chi_2$ in the sense of Definition 4.2.3; we write $(\underline{S}_1, \underline{\chi}_1) \cong_f (\underline{S}_2, \underline{\chi}_2)$.

4.2.5. Lemma Let X be a finite free abelian group. We have canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{cts}}(X \otimes \varprojlim_{E/F \text{ unramified}} \kappa_E^\times,\ \bar{\mathbb{F}}_p^\times) &= \varinjlim_{E/F \text{ unramified}} \mathrm{Hom}_{\mathrm{cts}}(X \otimes \kappa_E^\times,\ \bar{\mathbb{F}}_p^\times) \\ &= \varinjlim_{E/F \text{ unramified}} \mathrm{Hom}_{\mathrm{cts}}(X \otimes \mathcal{O}_E^\times,\ \bar{\mathbb{F}}_p^\times).\end{aligned}$$

Here transition maps are norm maps.

Proof. It follows from the Stone duality that in the category of profinite groups we have

$$\mathrm{Hom}_{\mathrm{cts}}(X \otimes \varprojlim_{E/F \text{ unramified}} \kappa_E^\times,\ \mathbb{F}_q^\times) = \varinjlim_{E/F \text{ unramified}} \mathrm{Hom}_{\mathrm{cts}}(X \otimes \kappa_E^\times,\ \mathbb{F}_q^\times).$$

Since $X \otimes \varprojlim_{E/F \text{ unramified}} \kappa_E^\times$ is a profinite group and is thus a compact object, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{cts}}(X \otimes \varprojlim_{E/F \text{ unramified}} \kappa_E^\times,\ \bar{\mathbb{F}}_p^\times) &= \varinjlim_q \mathrm{Hom}_{\mathrm{cts}}(X \otimes \varprojlim_{E/F \text{ unramified}} \kappa_E^\times,\ \mathbb{F}_q^\times) \\ &= \varinjlim_q \varinjlim_{E/F \text{ unramified}} \mathrm{Hom}_{\mathrm{cts}}(X \otimes \kappa_E^\times,\ \mathbb{F}_q^\times) \\ &= \varinjlim_{E/F \text{ unramified}} \varinjlim_q \mathrm{Hom}_{\mathrm{cts}}(X \otimes \kappa_E^\times,\ \mathbb{F}_q^\times) \\ &= \varinjlim_{E/F \text{ unramified}} \mathrm{Hom}_{\mathrm{cts}}(X \otimes \kappa_E^\times,\ \bar{\mathbb{F}}_p^\times)\end{aligned}$$

□

4.2.6. Definition Write $\widehat{\kappa_F^\times}^{\mathrm{Nm}}$ for

$$\varprojlim_{E/F \text{ unramified}} \kappa_E^\times,$$

and write $\widehat{\mathcal{O}_F^\times}^{\mathrm{Nm}}$ for

$$\varprojlim_{E/F \text{ unramified}} \mathcal{O}_E^\times.$$

4.2.7. Corollary Let $\underline{T}$ be a κ_F-torus.

(1) For each unramified extension E/F, the map

$$\mathrm{Hom}(\underline{T}(\kappa_F), \bar{\mathbb{F}}_p^\times) \xrightarrow{\chi \mapsto \chi \circ \mathrm{Nm}_{\kappa_E/\kappa_F}} \mathrm{Hom}(\underline{T}(\kappa_E), \bar{\mathbb{F}}_p^\times)$$

is an inclusion.

(2) We can identify $\mathrm{Hom}(\underline{T}(\kappa_F), \bar{\mathbb{F}}_p^\times)$ as a subgroup of $\mathrm{Hom}(X_*(\underline{T}) \otimes \widehat{\kappa_F^\times}^{\mathrm{Nm}}, \bar{\mathbb{F}}_p^\times)$. Under this identification, we have

$$\mathrm{Hom}(\underline{T}(\kappa_E), \bar{\mathbb{F}}_p^\times) = \mathrm{Hom}(X_*(\underline{T}) \otimes \widehat{\kappa_F^\times}^{\mathrm{Nm}}, \bar{\mathbb{F}}_p^\times)^{\mathrm{Gal}_{\kappa_F}}.$$

Proof. (1) Write $\widehat{\underline{T}}$ for the dual torus of $\underline{T}$. By Lemma 4.2.1, we have $\mathrm{Hom}(\underline{T}(\kappa_E), \bar{\mathbb{F}}_p^\times) \cong \widehat{\underline{T}}(\kappa_E)$ and $\mathrm{Hom}(\underline{T}(\kappa_F), \bar{\mathbb{F}}_p^\times) \cong \widehat{\underline{T}}(\kappa_F)$. It is clear that $\underline{T}(\kappa_F) = \underline{T}(\kappa_E)^{\mathrm{Gal}(\kappa_E/\kappa_F)}$.

(2) It follows from part (1) and Lemma 4.2.5. Note that for all E sufficiently large (splitting $\underline{T}$), $\underline{T}(\kappa_E) = X_*(uT) \otimes \kappa_E^\times$. □

4.2.8. Corollary Let T be an unramified F-torus, and write $T(F)^0$ for the Iwahori subgroup of $T(F)$. We have

$$\mathrm{Hom}(T(F)^0, \bar{\mathbb{F}}_p^\times) = \mathrm{Hom}(X_*(T) \otimes \widehat{\mathcal{O}_F^\times}^{\mathrm{Nm}}, \bar{\mathbb{F}}_p^\times)^{\mathrm{Gal}_F}.$$

Proof. It is equivalent to Corollary 4.2.7. □

4.2.9. Remark If we choose $T = \mathbb{G}_m$, then Corollary 4.2.8 together with the Hochschild-Serre spectral sequence immediately implies that the Artin repository map $\mathrm{Hom}(F^\times, \bar{\mathbb{F}}_p^\times) \to \mathrm{Hom}(\mathrm{Gal}_F, \bar{\mathbb{F}}_p^\times)$ is an isomorphism. The profinite group $\widehat{\mathcal{O}_F^\times}^{\mathrm{Nm}}$ can be identified with the abelianized inertia of Gal_F.

4.3. Inertial refinement of the mod p LLC for tame tori

In this subsection, we fix an F-tori T, with splitting field L. The local Langlands correspondence for tori holds for all divisible coefficients.

4.3.1. Theorem ([Ch20], [Yu09, 7.5]) If F is a local field, then there exists an isomorphism

$$\beta_T : H^1_{\mathrm{cts}}(W_{L/F}, X^*(T) \otimes D) \cong \mathrm{Hom}_{\mathrm{cts}}(T(F), D)$$

for any divisible abelian topological group D. Moreover, β_T is additive functorial in T (in the sense that β_T is an additive natural transformation between additive functors).

We will now fix a torsion divisible group D equipped with discrete topology.

4.3.2. Lemma We have $H^1_{\mathrm{cts}}(W_{L/F}, X^*(T)\otimes D) = H^1_{\mathrm{cts}}(\mathrm{Gal}_F, X^*(T)\otimes D)$.

Proof. It is clear since D-valued L-parameters have finite image. □

We also need to understand how the LLC for tori behaves under base change.

4.3.3. Proposition Let $E \subset F^s$ be a finite extension of F, and write $\mathrm{Nm}_{E/F} : E \to F$ for the norm map. There exists a commutative diagram

$$\begin{tikzcd} H^1_{\mathrm{cts}}(\mathrm{Gal}_F, X^*(T)\otimes D) \arrow[r] \arrow[d, "\rho:\mapsto\rho|_{\mathrm{Gal}_E}"] & \mathrm{Hom}_{\mathrm{cts}}(T(E), D) \arrow[d, "\chi:\mapsto\chi\circ\mathrm{Nm}_{E/F}"] \\ H^1_{\mathrm{cts}}(\mathrm{Gal}_E, X^*(T)\otimes D) \arrow[r] & \mathrm{Hom}_{\mathrm{cts}}(T(F), D) \end{tikzcd}$$

Proof. Let $T' := \mathrm{Res}_{E/F} T$. Let $f : T' \to T$ be the norm morphism. By unravelling definitions, f induces the restriction map $\rho \mapsto \rho|_{\mathrm{Gal}_E}$ for L-parameters, and induces the norm map $\chi \mapsto \chi \circ \mathrm{Nm}_{E/F}$ for characters of tori. The proposition is a special case of the functoriality of β_T. □

4.3.4. Definition Let $\rho \in H^1_{\mathrm{cts}}(\mathrm{Gal}_F, X^*(T)\otimes D)$ be an L-parameter. The *inertial type* of ρ is defined to be the image of ρ in $\varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}_{\mathrm{cts}}(T(E), D)$, and is denoted by $\beta_{T,I}(\rho)$.

We summarize basic results on integral models of tori as follows.

4.3.5. Proposition (1) There exists a unique maximal compact subgroup $T(F)^1 \subset T(F)$.

(2) We have

$$T(F)^1 = \{x \in T(F) | |\chi(x)|_p = 0 \text{ for all } \chi \in X^*(T)\}.$$

(3) Let $\mathscr{T}^{\mathrm{ft}}$ be the *ft-Néron model* of T. We have

$$T(F)^1 = \mathscr{T}^{\mathrm{ft}}(\mathcal{O}_F).$$

(4) Let $\mathscr{T}^0$ be the *connected Néron model* of T, and let $T(F)^0$ be the Iwahori subgroup of $T(F)$. We have

$$T(F)^0 = \mathscr{T}^0(\mathcal{O}_F).$$

(5) Let $T' \to T$ be a morphism of F-tori. Then $T'(F) \to T(F)$ maps $T'(F)^1$ to $T(F)^1$, and maps $T'(F)^0$ to $T(F)^0$,

(6) There exists a finite unramified extension E/F such that $T(E)/T(E)^0 = T(\breve{F})/T(\breve{F})^0$.

Proof. (1) and (2): See [KP22, Proposition 2.5.8].

(3): It is [KP22, Proposition B.7.2].

(3): It is [KP22, Proposition B.8.7].

(5): It is [KP22, Proposition 2.5.9] and [KP22, Proposition 2.5.19].

(6): It follows from [KP22, Corollary 11.1.6]. □

4.3.6. Lemma We have

$$\varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}_{\mathrm{cts}}(T(E), D) = \varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}_{\mathrm{cts}}(T(E)^0, D).$$

Proof. By part (6) of Proposition 4.3.5, there exists a finite unramified extension E/F such that $T(E)/T(E)^0 = T(\breve{F})/T(\breve{F})^0$.

Let F_1/F be a finite unramified extension containing E, and let E_1/F_1 be a finite unramified extension. It suffices to show for $\chi_1, \chi_2 : T(F_1) \to D$ such that $\chi_1|_{T(F_1)^0} = \chi_2|_{T(F_1)^0}$, $\chi_1 \circ \mathrm{Nm}_{E_1/F_1} = \chi_2 \circ \mathrm{Nm}_{E_1/F_1}$.

Write $\widetilde{\chi}_i = \chi_i \circ \mathrm{Nm}_{E/F}$, $i = 1, 2$. Suppose $\chi_1|_{T(F_1)^0} = \chi_2|_{T(F_1)^0}$. Let $x \in T(E_1)$ be an arbitrary element. Since $T(E_1)/T(E_1)^0 = T(F_1)/T(F_1)^0$, there exists $y \in T(F_1)$ such that $\frac{x}{y} \in T(E_1)^0$.

Part (5) of Proposition 4.3.5 applied to $\mathrm{Res}_{E_1/F_1}(T_{F_1}) \xrightarrow{\mathrm{Nm}_{E_1/F_1}} T_{F_1}$ shows that

$$\widetilde{\chi}_1|_{T(E_1)^0} = \widetilde{\chi}_2|_{T(E_1)^0}.$$

In particular,

$$\widetilde{\chi}_1(x/y) = \widetilde{\chi}_2(x/y).$$

Equivalently

$$\frac{\widetilde{\chi}_1(x)}{\widetilde{\chi}_2(x)} = \left(\frac{\chi_1(y)}{\chi_2(y)}\right)^{[E_1:F_1]}.$$

Since D is assumed to be a torsion divisible group, there exists an integer n such that $(\frac{\chi_1(y)}{\chi_2(y)})^n = 1$. Choose E_1 to be the unramified extension of F_1 of degree n inside F^s, and we have $\widetilde{\chi}_1(x) = \widetilde{\chi}_2(x)$. Since the group $T(F_1)$ is a finitely generated abelian group, we can choose E_1/F_1 such that $\widetilde{\chi}_1(x) = \widetilde{\chi}_2(x)$ for all $x \in T(E_1)$. □

4.3.7. Corollary If $D = \bar{\mathbb{F}}_p^\times$, we have

$$\varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}_{\mathrm{cts}}(T(E), D) = \varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D).$$

where κ_E is the residue field of E.

Proof. Combine Lemma 4.3.6 and part (4) of Proposition 4.3.5; also note that $\mathrm{Hom}(\mathscr{T}^0(\kappa_E), \bar{\mathbb{F}}_p^\times) = \mathrm{Hom}_{\mathrm{cts}}(\mathscr{T}^0(\mathcal{O}_E), \bar{\mathbb{F}}_p^\times)$. □

4.3.8. Lemma Assume $D = \bar{\mathbb{F}}_p^\times$. For each $\rho \in H^1_{\mathrm{cts}}(\mathrm{Gal}_F, \widehat{T}(\bar{\mathbb{F}}_p))$,

$$\beta_{T,I}(\rho) \in \mathrm{Im}(\mathrm{Hom}(\mathscr{T}^0(\kappa_F), D) \longrightarrow \varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D)).$$

Proof. We define a Gal_{κ_F}-action on $\varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D)$ as follows, for $\sigma \in \mathrm{Gal}_{\kappa_F}$ and $\chi : \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D))$, set $\sigma \cdot \chi := \chi \circ \sigma^{-1}$.

It is clear that $\beta_{T,I}(\rho)$ is a Gal_{κ_F}-fixed point. So it remains to show that

$$\begin{aligned}&\mathrm{Im}(\mathrm{Hom}(\mathscr{T}^0(\kappa_F), D) \to \varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D))\\ =&(\varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D))^{\mathrm{Gal}_{\kappa_F}}.\end{aligned}$$

The map above is clearly a well-defined injection. Let $\chi \in \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D)$ be a Gal_{κ_F}-invariant character. Write $\chi_0 \in \mathrm{Hom}(\mathscr{T}^0(\kappa_F), D)$ for the character $\frac{1}{[E:F]}\chi|_{\mathscr{T}^0(\kappa_F)}$ (note that D is a divisible group). It is clear that $\chi = \chi_0 \circ \mathrm{Nm}_{E/F}$. □

4.3.9. Proposition (1) There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{cts}}(T(F), \bar{\mathbb{F}}_p^\times) \arrow[r, "\beta_T^{-1}"] \arrow[d] & H^1_{\mathrm{cts}}(\mathrm{Gal}_F, \widehat{T}(\bar{\mathbb{F}}_p)) \arrow[d] \\
\mathrm{Hom}_{\mathrm{cts}}(T(F)^0, \bar{\mathbb{F}}_p^\times) \arrow[r, "\beta_{T,I}^{-1}"] & H^1_{\mathrm{cts}}(I_F, \widehat{T}(\bar{\mathbb{F}}_p))
\end{tikzcd}$$

where both vertical maps are restriction maps.

(2) Write $T_0 \subset T$ for the maximal unramified subtorus. All arrows in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{cts}}(T(F)^0, \bar{\mathbb{F}}_p^\times) \arrow[r, "\cong", "\beta_{T,I}^{-1}"'] \arrow[d, "\cong"] & H^1_{\mathrm{cts}}(I_F, \widehat{T}(\bar{\mathbb{F}}_p))^{\mathrm{Frob}} \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{cts}}(T_0(F)^0, \bar{\mathbb{F}}_p^\times) \arrow[r, "\cong", "\beta_{T,I}^{-1}"'] & H^1_{\mathrm{cts}}(I_F, \widehat{T}_0(\bar{\mathbb{F}}_p))^{\mathrm{Frob}}
\end{tikzcd}$$

are isomorphisms. Here $\mathrm{Frob} \in \mathrm{Gal}_F$ is a Frobenius element (that is, a topological generator of Gal_F modulo I_F).

Proof. (1) It suffices to show the composition

$$\mathrm{Hom}_{\mathrm{cts}}(T(F), \bar{\mathbb{F}}_p) \to \varinjlim_{E/F \text{ finite unramified}} \mathrm{Hom}(\mathscr{T}^0(\kappa_E), D)) \xrightarrow{\cong} H^1_{\mathrm{cts}}(I_F, \widehat{T}(\bar{\mathbb{F}}_p))$$

factors through $\mathrm{Hom}_{\mathrm{cts}}(T(F)^0, \bar{\mathbb{F}}_p)$, which is exactly the content of the Lemma 4.3.8.

(2) It suffices to show the vertical morphisms and the bottom horizontal morphism are isomorphisms.

The left vertical map is an isomorphism by [KP22, Corollary B.7.12]. The right vertical map is a special case of Proposition 2.5.3 (which will be elaborated in 4.5.4).

So it remains to analyze the bottom horizontal map. By local class field theory ([Iw86, 6.11]) and the fact that abelianization commutes with colimits, there is a canonical Frob-equivariant isomorphism

$$\widehat{\kappa_F^\times}^{\mathrm{Nm}} = \varprojlim_{E/F \text{ finite unramified}} \kappa_E^\times \cong I_F/P_F$$

where P_F is the wild inertia. Since all L-parameters we consider are tamely ramified, we have

$$H^1_{\mathrm{cts}}(I_F, \widehat{T}_0(\bar{\mathbb{F}}_p)) = H^1_{\mathrm{cts}}(I_F/P_F, \widehat{T}_0(\bar{\mathbb{F}}_p)) = H^1_{\mathrm{cts}}(\widehat{\kappa_F^\times}^{\mathrm{Nm}}, \widehat{T}_0(\bar{\mathbb{F}}_p)).$$

Since T_0 is an unramified torus, I_F acts on $\widehat{T}_0(\bar{\mathbb{F}}_p)$ trivially, and thus we have Frob-equivariant isomorphisms

$$\begin{aligned} H^1_{\mathrm{cts}}(\widehat{\kappa_F^\times}^{\mathrm{Nm}}, \widehat{T}_0(\bar{\mathbb{F}}_p)) &= \mathrm{Hom}_{\mathrm{cts}}(\widehat{\kappa_F^\times}^{\mathrm{Nm}}, \widehat{T}_0(\bar{\mathbb{F}}_p)) \\ &= \mathrm{Hom}_{\mathrm{cts}}(\widehat{\kappa_F^\times}^{\mathrm{Nm}}, \bar{\mathbb{F}}_p^\times \otimes X^*(T_0)) \\ &= \mathrm{Hom}_{\mathrm{cts}}(X_*(T_0) \otimes \widehat{\kappa_F^\times}^{\mathrm{Nm}}, \bar{\mathbb{F}}_p^\times). \end{aligned}$$

Write $\mathscr{T}^0$ for the (connected) Néron model of T^0, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{cts}}(T_0(F)^0, \bar{\mathbb{F}}_p^\times) &= \mathrm{Hom}_{\mathrm{cts}}(\mathscr{T}^0(\mathcal{O}_F), \bar{\mathbb{F}}_p^\times) \\ &= \mathrm{Hom}_{\mathrm{cts}}(X_*(T_0) \otimes \widehat{\kappa_F^\times}^{\mathrm{Nm}}, \bar{\mathbb{F}}_p^\times) \end{aligned}$$

where the last step is Corollary 4.2.7. □

4.4. Inertial refinement of the Deligne-Lusztig map

4.4.1. The restriction map Denote by $\mathfrak{SC}_G$ the set of stable conjugacy classes of Deligne-Lusztig data (S, χ) for G.

Let E/F be a finite unramified extension. There exists a map

$$\mathfrak{SC}_G \to \mathfrak{SC}_{G_E}$$

sending (S, χ) to $(S_E, \chi \circ \mathrm{Nm}_{E/F})$ where $\mathrm{Nm}_{E/F} : S(E) \to S(F)$ is the norm map.

Define

$$\mathfrak{SC}_{I_F,G} := \mathrm{Im}(\mathfrak{SC}_G \to \varinjlim_{E/F \text{ unramified}} \mathfrak{SC}_{G_E}).$$

Also denote by $\mathfrak{TI}_G \subset H^1(I_F, \widehat{G}(\bar{\mathbb{F}}_p))_{\mathrm{tame}}$ the set of tame inertial Langlands parameters $I_F \to {}^L G(\bar{\mathbb{F}}_p)$ that can be extended to Gal_F.

With the new notations introduced, Proposition 4.1.5 can be rephrased as the image of $\mathrm{DL} : \mathfrak{SC}_G \to H^1(\mathrm{Gal}_F, \widehat{G}(\bar{\mathbb{F}}_p))$ consists precisely the semisimple L-parameters.

4.4.2. Lemma $\mathrm{DL} : \mathfrak{SC}_{I_F,G} \to \mathfrak{TI}_G$ is surjective. So is $\varinjlim_{E/F \text{ unramified}} \mathfrak{SC}_{G_E} \to H^1(I_F, \widehat{G}(\bar{\mathbb{F}}_p))_{\mathrm{tame}}$.

Proof. By Lemma 3.2.9, a tame inertial type can be extended to a semisimple L-parameter. The lemma now follows from Proposition 4.1.5. □

To show $\mathrm{DL} : \mathfrak{SC}_{I_F,G} \to \mathfrak{TI}_G$ is actually a bijection, we will need to give a combinatorial description of both sets.

4.4.3. Parahorics For each maximal F-split torus S_s, we can attach an affine apartment $\mathcal{A}(S_s)$ of the Bruhat-Tits building $\mathcal{B}(G)$. For each vertex $x \in \mathcal{A}(S_s)$, a smooth group scheme $\mathcal{G}_x^\circ$ with generic fiber G and connected special fiber such that $\mathcal{G}_x^\circ(\mathcal{O}) \subset \mathrm{Fix}(x)$. There exists a closed $\mathcal{O}$-split torus $\mathcal{S}_s \subset \mathcal{G}_x^0$ with generic fiber S_s; the special fiber $\bar{\mathcal{S}}_s$ of $\mathcal{S}_s$ is a maximal κ_F-split torus in the special fiber of $\mathcal{G}_x^0$ ([KP22, 4.1.20]).

4.4.4. Notation Recall that we fixed an F-pinning $(B, T, \{X_\alpha\})$ of G. We can choose T so that it is maximally unramified and maximally split since G is quasi-split. Write $T_s \subset T$ for the maximal F-split subtorus and write $T_0 \subset T$ for the maximal unramified subtorus. The Chevalley valuation associated to the pinning $(B, T, \{X_\alpha\})$ determines a superspecial vertex ([Kal19a, Definition 3.4.8]) in the apartment $\mathcal{A}(T_s)$.

Let $\mathcal{T}_0$ be the maximal unramified $\mathcal{O}$-torus of $\mathcal{G}_x^\circ$ with generic fiber T_0. Write $\underline{T}$ for the special fiber of $\mathcal{T}_0$. Also write $\underline{G}$ for the reductive quotient of (the special fiber of) $\mathcal{G}_x^\circ$.

4.4.5. Definition An *inertial Deligne-Lusztig datum* is a pair $(\underline{S}, \underline{\chi})$ where $\underline{S}$ is a maximal κ_F-torus of $\underline{G}$ and $\underline{\chi} : \underline{S}(\kappa_F) \to \bar{\mathbb{F}}_p^\times$ is a character.

A *based inertial Deligne-Lusztig datum* is a pair $(\underline{S}, \underline{\chi}, \underline{B}_{\underline{S}})$ where $(\underline{S}, \underline{\chi})$ is an inertial Deligne-Lusztig datum and $\underline{B}_{\underline{S}} \subset \underline{G}$ is a Borel defined over $\bar{\mathbb{F}}_p$ containing $\underline{S}$.

Two inertial Deligne-Lusztig data $(\underline{S}, \underline{\chi})$ and $(\underline{S}', \underline{\chi}')$ are said to be *geometrically conjugate* if there exists an element $g \in \underline{G}(\bar{\mathbb{F}}_p)$ such that $(\underline{S}, \underline{\chi}) \cong_{\mathrm{Int}(g)} (\underline{S}', \underline{\chi}')$ in the sense of Definition 4.2.4.

Denote by $\mathfrak{S}\mathfrak{C}_{\underline{G}}$ the set of geometric conjugacy classes of inertial Deligne-Lusztig data.

4.4.6. Proposition There is a natural bijection

$$\varinjlim_{E/F \text{ unramified}} \mathfrak{S}\mathfrak{C}_{\underline{G}_{\kappa_E}} \to \varinjlim_{E/F \text{ unramified}} \mathfrak{S}\mathfrak{C}_{G_E}.$$

Proof. Let $(\underline{S}, \underline{\chi}) \in \mathfrak{S}\mathfrak{C}_{\underline{G}}$ be an inertial Deligne-Lusztig datum. By [KP22, Proposition 8.2.1 (1)], there exists a closed unramified $\mathcal{O}$-torus $\mathcal{S}_0$ of $\mathcal{G}_x^0$ whose special fiber is $\underline{S}$. The generic fiber S_0 of $\mathcal{S}_0$ is a maximal unramified torus of G and the centralizer S of S_0 in G is a maximally unramified maximal F-torus. Write χ_0^0 for the inflated character

$$\chi_0^0 : S_0(F)^0 \to \underline{S}(\kappa_F) \xrightarrow{\chi_0} \bar{\mathbb{F}}_p^\times.$$

By part (2) of Proposition 4.3.9, the character χ_0^0 can be extend uniquely to a character $\chi^0 : S(F)^0 \to \bar{\mathbb{F}}_p^\times$. Since $\bar{\mathbb{F}}_p^\times$ is a divisible group, $\mathrm{Hom}(-, \bar{\mathbb{F}}_p^\times)$ is exact, the character χ^0 can be extended to $\chi : S(F) \to \bar{\mathbb{F}}_p^\times$. The pair (S, χ) is a Deligne-Lusztig datum.

We first show the well-definedness. Suppose $(\underline{S}, \underline{\chi})$ and $(\underline{S}', \underline{\chi}')$ are geometrically conjugate. By [KP22, Proposition 8.2.1 (5)], there exists an element $g \in \mathcal{G}_x^\circ(\mathcal{O}_{\breve{F}}) \subset G(\breve{F})$ such that $gS_0g^{-1} = S_0'$ and $\mathrm{Int}(g)_*\underline{\chi} \cong \underline{\chi}'$. Since S (resp. S') is the centralizer of S_0 (resp. S_0'), we have $gSg^{-1} = S'$. Let E/F be an unramified extension such that $g \in G(E)$ and both $\underline{S}_{\kappa_E}, \underline{S}'_{\kappa_E}$ are split. Then we have $\mathrm{Int}(g)_*\underline{\chi} = \underline{\chi}'$ and $gS(E)g^{-1} = S(E)'$. By Corollary 4.3.7, there exists a finite unramified extension E'/E such that $\mathrm{Int}(g)_*\underline{\chi} \circ \mathrm{Nm}_{E'/E} = \underline{\chi}' \circ \mathrm{Nm}_{E'/E}$. As a consequence, (S, χ) and (S', χ') are stably conjugate after restricting to E'. Corollary 4.3.7 also ensures the map is injective. Suppose (S, χ) and (S', χ') are stably conjugate after restricting to some E. Since the apartment of S and of S' both contains the vertex x, by [Kal19a, Lemma 3.4.12], there exists an element $g \in \mathcal{G}_x^\circ(\mathcal{O}_{\breve{F}})$ such that $gSg^{-1} = S'$. Corollary 4.3.7 implies $(\underline{S}, \underline{\chi})$ and $(\underline{S}', \underline{\chi}')$ are geometrically conjugate (under the reduction $\bar{g}$ of g) after restricting to some finite unramified extension E'/E.

Next, we show the surjectivity of the map. Let S, $\underline{S}$ and S_0 be as in the first paragraph of the proof, and let (S', χ') be an arbitrary Deligne-Lusztig datum. Since both S and S' are both maximally unramified (i.e. they become split after base change to $\breve{F}$), there exists an element $g \in G(\breve{F})$ such that $gS'g^{-1} = S$. We may assume $g \in G(E)$ for some finite unramified extension E/F. Write

χ_1 for $\mathrm{Int}(g)_*\chi'$. The restriction to $S_0(E)^0$ if χ_1 factors through $\underline{\chi}_1 : \underline{S}(\kappa_E) \to \bar{\mathbb{F}}_p^\times$. The inertia Deligne-Lusztig datum $(\underline{S}_{\kappa_E}, \underline{\chi}_1)$ is mapped to the equivalence class of (S', χ'). $\square$

4.4.7. Root system for the reductive quotient of a superspecial parahoric The relative root system $\Phi_{\breve{F}}(T_0, G)$ of G with respect to T_0 is not reduced in general. The absolute root system $\Phi_{\bar{\mathbb{F}}_p}(\underline{T}, \underline{G})$ of $\underline{G}$ is a reduced modification of the possibly non-reduced root system $\Phi_{\breve{F}}(T_0, G)$ (see [KP22, Proposition 8.4.8]). In particular, $\Phi_{\breve{F}}(T_0, G)$ and $\Phi_{\bar{\mathbb{F}}_p}(\underline{T}, \underline{G})$ have the same Weyl group (see [Ha18, Remark 3.4]), and their Weyl group can be identified with the I_F-invariant subgroup (which we denote by Ω^θ) of the absolute Weyl group $\Omega := N_{T(F^s)}(G(F^s))/T(F^s)$ ([Ha15, Lemma 4.2]).

4.4.8. Theorem (1) Let E/F be a finite unramified extension. The restriction functor

$$\mathfrak{S}\mathfrak{C}_{\underline{G}} \to \mathfrak{S}\mathfrak{C}_{\underline{G}_{\kappa_E}}$$

sending $(\underline{S}, \underline{\chi})$ to $(\underline{S}_{\kappa_E}, \underline{\chi} \circ \mathrm{Nm}_{\kappa_E/\kappa_F})$ is injective.

(2) The set $\mathfrak{S}\mathfrak{C}_{\underline{G}}$ is in natural bijection with

$$((X^*(\underline{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z})/\Omega^\theta)^{\mathrm{Frob}} \cong ((X^*(\underline{T}) \otimes \bar{\mathbb{F}}_p^\times)/\Omega^\theta)^{\mathrm{Frob}}$$

where Frob is the Frobenius map on $\underline{T}$ corresponding to the standard κ_F-rational structure.

(3) The set $\varinjlim_{E/F \text{ unramified}} \mathfrak{S}\mathfrak{C}_{\underline{G}_{\kappa_E}}$ is in natural bijection with

$$(X^*(\underline{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z})/\Omega^\theta \cong (X^*(\underline{T}) \otimes \bar{\mathbb{F}}_p^\times)/\Omega^\theta.$$

Proof. (1) It is [DL76, Proposition 5.4].

(2, 3) It is [DL76, Proposition 5.7]. $\square$

4.4.9. Summary We have established the following diagram:

$$\begin{tikzcd}
\mathfrak{S}\mathfrak{C}_{\underline{G}} \arrow[r, hook] \arrow[d, hook] & \mathfrak{S}\mathfrak{C}_{I_F,G} \arrow[r, two heads] \arrow[d, hook] & \mathfrak{T}\mathfrak{I}_G \arrow[d, hook] \\
\varinjlim_{E/F \text{ unramified}} \mathfrak{S}\mathfrak{C}_{\underline{G}_{\kappa_E}} \arrow[r, equal] \arrow[d, equal] & \varinjlim_{E/F \text{ unramified}} \mathfrak{S}\mathfrak{C}_{G_E} \arrow[r, two heads] & H^1(I_F, \widehat{G}(\bar{\mathbb{F}}_p))_{\mathrm{tame}} \\
(X^*(\underline{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z})/\Omega^\theta & &
\end{tikzcd}$$

4.5. Tame types and semisimple L-parameters

Let $q := p^f$ be a p-power integer. Denote by $\mathrm{Frob}_q : x \mapsto x^q$ the relative q-Frobenius map.

In the rest of this subsection, the dual groups $\widehat{G}$ and ${}^L G_{\bar{\mathbb{F}}_p}$ are always defined over $\bar{\mathbb{F}}_p$. Write L for the splitting field of G. We regard $\mathrm{Gal}(L/F)$ as a subgroup of ${}^L G_{\bar{\mathbb{F}}_p} = \widehat{G} \rtimes \mathrm{Gal}(L/F)$. Fix a Frobenius element $\sigma \in \mathrm{Gal}_F$ and denote its image in ${}^L G_{\bar{\mathbb{F}}_p}$ and $\mathrm{Gal}(L/F)$ by $\bar{\sigma}$; also fix a generator θ of the tame inertia of Gal_F and denote its image in ${}^L G_{\bar{\mathbb{F}}_p}$ and $\mathrm{Gal}(L/F)$ by $\bar{\theta}$. Denote by e the ramification index of $\mathrm{Gal}(L/F)$.

Define the *twisted q-Frobenius map*

$$F^\sigma_\theta : \widehat{G} \to \widehat{G}, \quad g \mapsto \bar{\sigma}^{-1}(\prod_{i=0}^{q-1} \bar{\theta}^i g \bar{\theta}^{-i})\bar{\sigma}.$$

4.5.1. Definition A semisimple $\bar{\theta}$-twisted conjugacy class $[s]_{\bar{\theta}}$ of $\widehat{G}$ is said to be F^σ_θ*-stable* if $s \in [s]_{\bar{\theta}}$ implies $F^\sigma_\theta(s) \in [s]_{\bar{\theta}}$.

4.5.2. Lemma A semisimple $\bar{\theta}$-twisted conjugacy class $[s]_{\bar{\theta}}$ is F^σ_θ-stable if and only if for each representative s of $[s]_{\bar{\theta}}$, there exists a tamely ramified L-parameter $\rho : \mathrm{Gal}_F \to {}^LG_{\bar{\mathbb{F}}_p}$ such that $\rho(\theta) = s\bar{\theta}$.

Proof. Unravel the definitions. □

4.5.3. Corollary A semisimple $\bar{\theta}$-twisted conjugacy class $[s]_{\bar{\theta}}$ is F^σ_θ-stable if and only if for some representative s of $[s]_{\bar{\theta}}$, there exists a semisimple L-parameter $\rho : \mathrm{Gal}_F \to {}^LG_{\bar{\mathbb{F}}_p}$ such that $\rho(\theta) = s\bar{\theta}$.

Proof. Combine Lemma 4.5.2 and Lemma 3.2.9. □

4.5.4. Proposition (1) The map F^σ_θ is a θ-twisted Frobenius endomorphism in the sense of 2.5.2.

(2) There is a one-to-one correspondence between $\bar{\theta}$-twisted semisimple conjugacy classes in $\widehat{G}$ and $\widehat{G}$-conjugacy classes of L-parameters $I_F \to {}^LG_{\bar{\mathbb{F}}_p}$, given by $[s]_{\bar{\theta}} \mapsto (\theta \mapsto s\bar{\theta})$.

(3) There is a one-to-one correspondence between F^σ_θ-stable $\bar{\theta}$-twisted semisimple conjugacy classes in $\widehat{G}$ and $\widehat{G}$-conjugacy classes of L-parameters $I_F \to {}^LG_{\bar{\mathbb{F}}_p}$ that can be extended to a semisimple L-parameter $\rho : \mathrm{Gal}_F \to {}^LG_{\bar{\mathbb{F}}_p}$.

Proof. (1), (2): Unravel the definitions.
(3): It is Corollary 4.5.3. □

4.5.5. Corollary In the context of Paragraph 4.4.9, there are natural bijections

$$\mathfrak{SC}_{\underline{G}} \cong \mathfrak{SC}_{I_F,G} \cong \mathfrak{TJ}_G.$$

Proof. By Paragraph 4.4.9, it suffices to show $\mathfrak{SC}_{\underline{G}} \to \mathfrak{TJ}_G$ is bijective. By Theorem 4.4.8, we have $\mathfrak{SC}_{\underline{G}} \cong ((X^*(\underline{T}) \otimes \bar{\mathbb{F}}_p^\times)/\Omega^\theta)^{\mathrm{Frob}}$. By Proposition 2.5.3, we have $\mathfrak{TJ}_G \cong (X_*(\widehat{T})_{\theta,\mathrm{tf}} \otimes \bar{\mathbb{F}}_p^\times/\Omega^\theta)^{\varphi\otimes\mathrm{Frob}_q}$. It remains to show $X_*(\widehat{T})_{\theta,\mathrm{tf}}$ and $X^*(\underline{T}) = X^*(T_0)$ are canonically identified. Note that $X_*(\widehat{T})_{\theta,\mathrm{tf}}$ is by definition the maximal unramified finite free quotient of $X_*(\widehat{T})$. Since $X_*(T_0)$ is the maximal unramified subgroup of $X_*(T)$, by duality, $X^*(T_0)$ is the maximal unramified finite free quotient of $X^*(T)$. Since $X^*(T) = X_*(\widehat{T})$, $X_*(\widehat{T})_{\theta,\mathrm{tf}}$ and $X^*(\underline{T})$ are unramified. □

5. Digression: de Rham lifts of semisimple mod p L-parameters of regular Hodge type

Contents

To study the Emerton-Gee stacks, it is important to construct regular de Rham lifts of all mod p L-parameters.

For semisimple L-parameters, such lifts can be easily constructed via the Langlands-Shelstad factorization. Recall that if $S \subset G$ is a maximally unramified F-torus, then there exists an L-embedding ${}^Lj : {}^LS(\bar{\mathbb{Z}}_p) \to {}^LG(\bar{\mathbb{Z}}_p)$ (see the remarks after Theorem 3.3.10).

5.1. Basic facts about de Rham L-parameters

Let $\Lambda \supset \mathbb{Z}_p$ be a discrete valuation ring.

5.1.1. Definition An L-parameter $\rho : \mathrm{Gal}_F \to {}^LG(\Lambda)$ is said to be semistable (resp. crystalline) if for some closed embedding of algebraic groups ${}^LG \hookrightarrow \mathrm{GL}_d$ the composite $\mathrm{Gal}_F \to \mathrm{GL}_d(\Lambda)$ is semistable (resp. crystalline).

An L-parameter $\rho : \mathrm{Gal}_F \to {}^LG(\Lambda)$ is said to be potentially semistable (resp. crystalline) if there exists a finite extension E/F such that $\rho|_{\mathrm{Gal}_E}$ is semistable (resp. crystalline).

5.1.2. Lemma An L-parameter $\rho : \mathrm{Gal}_F \to {}^LG(\Lambda)$ is semistable (resp. crystalline) if and only if its restriction to inertia $\rho|_{I_F} : I_F \to {}^LG(\Lambda)$ is semistable (resp. crystalline). Here we regard I_F as the absolute Galois group of the strict henselization $\breve{F}$ of F.

Proof. It is [BC08, Proposition 9.3.1]. □

5.1.3. Hodge-Tate theory The main reference is [L22, Section 5] and [BG19, Section 2.8]. Write E for $\Lambda[1/p]$. Let $\rho : \mathrm{Gal}_F \to {}^LG(E)$ be a potentially semistable L-parameter. Then we can associate to ρ a tuple of cocharacter $\mathrm{HT}^\iota(\rho)$ of $\widehat{G}_{\mathbb{C}}$ for each embedding $\iota : E \hookrightarrow \mathbb{C}$. For each embedding $f : {}^LG \to {}^LH$, we have $\mathrm{HT}^\iota(f \circ \rho) = f \circ \mathrm{HT}^\iota(\rho)$.

We can regard $\mathrm{HT}_{\mathbb{C}}(\rho) := \boxtimes_{\iota : E \hookrightarrow \mathbb{C}} \mathrm{HT}^\iota(\rho)$ as a cocharacter of $\prod_\iota \widehat{G}_{\mathbb{C}} \cong \mathrm{Res}_{\mathbb{C}\otimes_{\mathbb{Q}_p} E/\mathbb{C}} \widehat{G}$. By Tannakian formalism, a cocharacter is equivalent to an exact tensor grading. Write $\mathrm{dR}_{\mathbb{C}}(\rho)$ for the canonical exact tensor filtration associated to $\mathrm{HT}_{\mathbb{C}}(\rho)$. When ρ is potentially semistable (say $\rho|_{\mathrm{Gal}_{F'}}$ is semistable), then ρ corresponds to a filtered $(\varphi, N, \mathrm{Gal}(F'/F))$-module D_0 with ${}^LG$-structure. Write $F_0' \subset F'$ for the maximal subfield unramified over $\mathbb{Q}_p$. Note that $D := D_0 \otimes_{F_0'} F'$ is an ${}^LG$-torsor over $F' \otimes_{\mathbb{Q}_p} E$ equipped with a $\mathrm{Gal}(F'/F)$-stable exact tensor filtration $\mathrm{dR}_{F'}(\rho)$ which recovers $\mathrm{dR}_{\mathbb{C}}(\rho)$ after base change to $\mathbb{C} \otimes_{\mathbb{Q}_p} E$. On the other hand, D descends to an ${}^LG$-torsor over D_F over $F \otimes_{\mathbb{Q}_p} E$ and $\mathrm{dR}_{F'}(\rho)$ descends to an exact tensor filtration $\mathrm{dR}_F(\rho)$ on D_F. Following [BG19], we will call $\mathrm{dR}_F(\rho)$ the *Hodge type* of ρ. The potentially semistable L-parameter ρ is said to be *of regular Hodge type* or *Hodge regular* if the stabilizer of $\mathrm{dR}_F(\rho)$ is a Borel subgroup of (a form of) $\mathrm{Res}_{F\otimes_{\mathbb{Q}_p} E/E} {}^LG$. Since the property of being a Borel subgroup descends along field extensions, we see ρ is Hodge regular if and only if each $\mathrm{HT}^\iota(\rho)$ is a regular cocharacter in the cocharacter lattice $X_*(\widehat{T})$ (here $\widehat{T}$ is a maximal torus of $\widehat{G}$ containing the image of $\mathrm{HT}^\iota(\rho)$), that is, $\mathrm{HT}^\iota(\rho)$ is not killed by any root of $\widehat{G}$ with respect to $\widehat{T}$.

5.2. The p-adic Hodge theoretic refinement of the LLC for tori

The results in this subsection is standard. The main reference is [Se89].

Let T be an F-torus which splits over L. Fix an algebraic closure $\bar{\mathbb{Q}}_p$ and let $E \subset \bar{\mathbb{Q}}_p$ be a finite extension of L.

Since $\bar{\mathbb{Z}}_p^\times$ is a divisible abelian group, by 4.3.1, there exists a functorial isomorphism

$$\beta_T : H^1_{\mathrm{cts}}(W_{L/F}, X^*(T) \otimes \bar{\mathbb{Z}}_p^\times) \cong \mathrm{Hom}_{\mathrm{cts}}(T(F), \bar{\mathbb{Z}}_p^\times).$$

Since we are working with integral coefficients, the Galois form of L-parameters and the Weil form of L-parameters are equivalent, so $H^1_{\mathrm{cts}}(W_{L/F}, X^*(T) \otimes \bar{\mathbb{Z}}_p^\times) = H^1_{\mathrm{cts}}(\mathrm{Gal}_F, \widehat{T}(\bar{\mathbb{Z}}_p))$ (see 3.1.2).

5.2.1. Locally algebraic characters A character $\chi \in \mathrm{Hom}_{\mathrm{cts}}(T(F), E^\times)$ is said to be *algebraic* if there exists an algebraic character $\psi \in \mathrm{Hom}(\mathrm{Res}_{F/\mathbb{Q}_p} T, \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m)$ such that $\chi = \psi(\mathbb{Q}_p) : T(F) \to \mathbb{G}_m(E)$.

A character $\chi \in \mathrm{Hom}_{\mathrm{cts}}(T(F), E^\times)$ is said to be *locally algebraic* if χ coincides with some algebraic character in an open neighborhood of 1.

Facts (See, for example, [C11, Appendix B]) χ is locally algebraic if and only if $\beta_T^{-1}(\chi)$ is Hodge-Tate, if and only if $\beta_T^{-1}(\chi)$ is potentially semistable, and if and only if $\beta_T^{-1}(\chi)$ is potentially crystalline.

Notation Write $\mathrm{Hom}_{\mathrm{cts}}(T(F), -)_{\mathrm{l.alg.}}$ for the locally algebraic subgroup of $\mathrm{Hom}_{\mathrm{cts}}(T(F), -)$, and write $H^1_{\mathrm{cts}}(\mathrm{Gal}_F, X^*(T) \otimes -)_{\mathrm{HT}}$ for the Hodge-Tate subset of $H^1_{\mathrm{cts}}(\mathrm{Gal}_F, X^*(T) \otimes -)$.

5.2.2. Lemma The composition

$$\mathrm{Hom}_{\mathrm{cts}}(T(F), \bar{\mathbb{Z}}_p^\times)_{\mathrm{l.alg.}} \xrightarrow{\beta_T^{-1}} H^1_{\mathrm{cts}}(\mathrm{Gal}_F, X^*(T) \otimes \bar{\mathbb{Z}}_p^\times)_{\mathrm{HT}} \xrightarrow{\mathrm{HT}} \prod_{\iota : \bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}} X_*(T)$$

is a group homomorphism.

Proof. Since the formation of co-labelled Hodge-Tate cocharacter is insensitive to restriction of Galois groups, the general case is reduced to the split tori case, which is clear. ☐

5.2.3. Definition Denote by

$$\mathfrak{H}^\iota : \mathrm{Hom}_{\mathrm{cts}}(T(F), \bar{\mathbb{Z}}_p^\times)_{\mathrm{l.alg.}} \xrightarrow{\mathrm{HT} \circ \beta_T^{-1}} \prod_{\iota : \bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}} X_*(T) \xrightarrow{\iota\text{-th component}} X_*(T)$$

the group homomorphism attaching to a character of $T(F)$ its ι-colabelled Hodge-Tate cocharacter.

5.2.4. Lubin-Tate Galois characters Write $\iota_0 : E \to \mathbb{C}$ for the distinguished embedding (recall that E is a subfield of the fixed algebraic closure $\bar{\mathbb{Q}}_p$).

The identity map $(\mathbb{G}_m)_E \xrightarrow{=} (\mathbb{G}_m)_E$ induces a $E^\times$-valued character

$$\chi_{\mathrm{LT}} : \mathbb{G}_m(E) \to E^\times = \mathbb{G}_m(E)$$

of $\mathbb{G}_m(E)$. Under the Local Langlands for split tori (= Local Class Field Theory), the associated L-parameter $\rho_{\mathrm{LT}} : \mathrm{Gal}_E \to E^\times$ is the so-called *Lubin-Tate Galois character*.

The Hodge-Tate weights for Lubin-Tate characters are computed by Serre. We have $\mathrm{HT}^\iota(\rho_{\mathrm{LT}}) = 0$ if $\iota \neq \iota_0$ and $\mathrm{HT}^{\iota_0}(\rho_{\mathrm{LT}}) = -1$ (see, for example, [L22, Appendix A.3]). Here we naturally label the cocharacters of $\mathbb{G}_m$ by integers.

The Lubin-Tate character satisfies $\rho_{\mathrm{LT}}(I_E) \subset \mathcal{O}_E^\times$. If we choose a uniformizer ϖ_E of E, and let $\chi_\varpi : \mathbb{G}_m(E) \to \mathcal{O}_E^\times$ be the character which sends ϖ_E to 1 and agrees with χ_{LT} on $\mathcal{O}_E^\times$. Then χ_ϖ corresponds to an integral Galois character $\rho_\varpi : \mathrm{Gal}_E \to \mathcal{O}_E^\times$, which is usually called *the fundamental character*.

5.2.5. Construction of crystalline Galois characters

Let $x \in X_*(\widehat{T}) = X^*(T) \cong \mathbb{Z}^{\oplus \dim_F T}$ be an arbitrary cocharacter.

Since T splits over L, we have $T(L) = (L^\times)^{\times \dim_F T}$. Write $\widetilde{\chi}_x : T(L) \to \mathcal{O}_L^\times$ be the locally algebraic character which is the composition of x and the fundamental character. Set $\chi_x := \widetilde{\chi}_x|_{T(F)}$. Note that $\mathrm{Gal}(L/F)$ acts on $X_*(T)$.

5.2.6. Lemma Assume L/F is a Galois extension. We have $\mathfrak{H}^{\iota_0\circ\theta}(\chi_x) = -\theta^{-1}\cdot x$ for all $\theta \in \mathrm{Gal}(L/F)$.

Proof. Since the formation of co-labelled Hodge-Tate cocharacter is insensitive to restriction of field, it suffices to compute the co-labelled Hodge-Tate cocharacters of the composite

$$\mathrm{Res}_{L/F} T(F) \xrightarrow{\mathrm{Nm}_{L/F}} T(F) \xrightarrow{\chi_x} L^\times.$$

Note that

$$\chi_x \circ \mathrm{Nm}_{L/F} = \prod_{\sigma\in\mathrm{Gal}(L/F)} \widetilde{\chi}_x \circ \sigma.$$

We have

$$\begin{aligned}
\mathfrak{H}^{\iota_0\circ\theta}(\chi_x \circ \mathrm{Nm}_{L/F}) &= \sum_{\sigma\in\mathrm{Gal}(L/F)} \mathfrak{H}^{\iota_0\circ\theta}(\widetilde{\chi}_x \circ \sigma) \\
&= \sum_{\sigma\in\mathrm{Gal}(L/F)} \mathfrak{H}^{\iota_0\circ\theta}(\sigma \circ \widetilde{\chi}_{\sigma^{-1}\cdot x}) \\
&= \sum_{\theta\in\mathrm{Gal}(L/F)} \mathfrak{H}^{\iota_0\circ\theta}(\sigma \circ \widetilde{\chi}_{\sigma^{-1}\cdot x}) \\
&= \sum_{\theta\in\mathrm{Gal}(L/F)} \mathfrak{H}^{\iota_0\circ(\theta\sigma^{-1})}(\widetilde{\chi}_{\sigma^{-1}\cdot x}) \\
&= \sum_{\theta\in\mathrm{Gal}(L/F)} \begin{cases} 0 & \theta\sigma^{-1} \neq 1 \\ -\sigma^{-1}\cdot x & \theta\sigma^{-1} = 1 \end{cases} \\
&= -\theta^{-1}\cdot x.
\end{aligned}$$

The fourth equality follows from [L22, Corollary 4, Appendix A.2], and the fifth equality follows from the Lubin-Tate Galois character computation (5.2.4). ∎

5.3. Existence of de Rham lifts of prescribed Hodge types

5.3.1. Theorem Let G be a quasi-split tame group over F, and let $\bar{\rho} : \mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$ be a semisimple mod p L-parameter.

There exists a maximal F-torus S of G such that for each cocharacter $x \in X_*(\widehat{S}) \subset X_*(\widehat{G})$, there exists a potentially crystalline lift $\rho : \mathrm{Gal}_F \to {}^LG(E)$ of $\bar{\rho}$ such that $\mathrm{HT}^{\iota_0\circ\theta}(\rho) = -\theta^{-1}\cdot x$ for all $\theta \in \mathrm{Gal}(E/F)$. Here E is a sufficiently large extension of F containing L.

In particular, $\bar{\rho}$ admits a de Rham lift of regular Hodge type.

Proof. By Theorem 3.4.1, $\bar{\rho}$ factors through ${}^LS(\bar{\mathbb{F}}_p)$ for some maximally unramified torus S of G. By possibly enlarging L, we assume L is a splitting field of S.

By the LLC for tori, $\bar{\rho}$ corresponds to a character $\chi_{\bar{\rho}} : S(F) \to \bar{\mathbb{F}}_p^\times$, which admits a Teichmüller lift $[\chi_{\bar{\rho}}] : S(F) \to W(\bar{\mathbb{F}}_p)^\times$. Applying the LLC for tori once again, $[\chi_{\bar{\rho}}] : S(F) \to W(\bar{\mathbb{F}}_p)^\times \subset \bar{\mathbb{Z}}_p^\times$ corresponds to a finite image L-parameter $[\bar{\rho}] : \mathrm{Gal}_F \to {}^LS(\bar{\mathbb{Z}}_p)$. The lift $[\bar{\rho}]$ is potentially crystalline of trivial Hodge type.

It remains to modify $[\bar{\rho}]$ so that it has the desired Hodge type. By Lemma 5.2.6, there exists a locally algebraic character $\chi_x : S(F) \to \mathcal{O}_L^\times$ such that $\mathfrak{H}^{\iota_0 \circ \theta}(\chi_x) = -\theta^{-1} \cdot x$ for all $\theta \in \mathrm{Gal}(L/F)$. Write $\bar{\chi}_x : S(F) \to \bar{\mathbb{F}}_p^\times$ for the reduction mod p of χ_x. The product

$$\chi := \chi_x[\bar{\chi}_x^{-1}\chi_{\bar{\rho}}] : S(F) \to (LW(\bar{\mathbb{F}}_p))^\times$$

is a locally algebraic character. By the LLC for tori, χ corresponds to an L-parameter $\rho : \mathrm{Gal}_F \to {}^LS(\mathcal{O}_E)$ lifting $\bar{\rho}$. Here E is a sufficiently large coefficient field.

To show ρ can be made Hodge regular, by the discussion in 5.1.3, it suffices to ensure $\theta^{-1} \cdot x$ is a regular cocharacter for all θ. Irregular cocharacters lie on the wall of Weyl chambers. The $\mathrm{Gal}(L/F)$-orbit of irregular cocharacters is contained in a finite union of hyperplanes of $X_*(\widehat{S}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since it is impossible to cover all integral points of $\mathbb{R}^d$ using a finite number of hyperplanes, there is a choice of x which makes ρ Hodge regular. □

6. Parahoric Serre weights

Contents

6.1. Algebraic representations

6.1.1. Simple $\underline{G}$-modules Let $\underline{G}$ be a connected reductive group over $\bar{\mathbb{F}}_p$.

Let $(\underline{B}, \underline{T})$ be a Borel pair of $\underline{G}$. For each dominant character $\lambda \in X^*(\underline{T})$, there exists a simple $\underline{G}$-module $L(\lambda)$ of highest weight λ. Any simple $\underline{G}$-module is isomorphic to exactly one such $L(\lambda)$ ([Jan03, II.2.4]).

Let $(\underline{B}', \underline{T}')$ be another Borel pair of $\underline{G}$. There exists an element $g \in \underline{G}$ such that $(\underline{B}', \underline{T}') = g(\underline{B}, \underline{T})g^{-1}$. Then a simple $\underline{G}$-module V has highest weight λ with respect to $(\underline{B}, \underline{T})$ if and only if it has highest weight $\lambda \circ \mathrm{Int}(g^{-1})$ with respect to $(\underline{B}', \underline{T}')$.

6.1.2. Simple $\underline{G}^{\mathbf{F}}$-modules Suppose the derived subgroup of $\underline{G}$ is simply-connected. Equip $\underline{G}$ with a $\mathbb{F}_{p^r}$-structure. Since all reductive groups over a finite field is quasi-split, specifying a $\mathbb{F}_{p^r}$-structure amounts to specifying a finite order automorphism π of the based root datum of $\underline{G}$. Such a π induces a Frobenius map $\mathbf{F} : \underline{G} \to \underline{G}$. Fix a $\mathbf{F}$-stable Borel pair $(\underline{B}, \underline{T})$. Simple $\underline{G}^{\mathbf{F}}$-modules arise from restrictions of simple $\underline{G}$-modules ([Her09, Appendix A]). Write

$$X_r(\underline{T}) := \{\lambda \in X^*(\underline{T}) | 0 \le \langle \lambda, \alpha^\vee \rangle < p^r, \ \forall \alpha \in \Delta(\underline{B}, \underline{T})\}$$

and

$$X^0(\underline{T}) := \{\lambda \in X^*(\underline{T}) | \langle \lambda, \alpha^\vee \rangle = 0, \ \forall \alpha \in R(\underline{B}, \underline{T})\}.$$

Then we have a bijection ([Her09, Proposition A.1.3])

$$\frac{X_r(\underline{T})}{(p^r-\pi)X^0(\underline{T})} \xrightarrow{\cong} \{\text{Irreducible representations of } \underline{G}(\mathbb{F}_{p^r}) = \underline{G}^{\mathbf{F}}\}$$
$$\lambda \mapsto L(\lambda)|_{\underline{G}^{\mathbf{F}}}$$

6.1.3. Based inertial Deligne-Lusztig data and simple $\underline{G}^{\mathbf{F}}$-modules Recall that a based inertial Deligne-Lusztig datum is a tuple $(\underline{S}, \underline{\chi}, \underline{B}_{\underline{S}})$ (Definition 4.4.5). Recall that there exists a short exact sequence (Equation (3))

$$0 \to X^*(\underline{S}) \xrightarrow{p^r-\pi} X^*(\underline{S}) \xrightarrow{\Xi} \mathrm{Hom}(\underline{S}^{\mathbf{F}}, \bar{\mathbb{F}}_p^\times) \to 0.$$

6.1.4. Lemma Let $(\underline{S}, \underline{\chi}, \underline{B}_{\underline{S}})$ be a based inertial Deligne-Lusztig datum. If $\underline{B}_{\underline{S}}$ is $\mathbf{F}^m$-stable, then the map

$$X_{rm}(\underline{S}) \to \frac{X^*(\underline{S})}{(p^{rm}-\pi^m)X^*(\underline{S})}$$

is surjective.

Proof. It is harmless to assume $m=1$. Let $\{\alpha\} = \Delta := \Delta(\underline{B}_{\underline{S}}, \underline{S})$ be the set of simple roots with respect to $\underline{B}_{\underline{S}}$. Write $\{\alpha^\vee\}_{\alpha\in\Delta}$ for the set of coroots, and write $\{\omega_\alpha\}_{\alpha\in\Delta} \subset X^*(\underline{S})$ for the fundamental weights; they form a basis of the weight lattice that is dual to $\{\alpha^\vee\}$.

We can replace $\underline{G}$ by its derived subgroup and thus assume $\underline{G}$ is semisimple. Since $\underline{G}$ is a simply-connected semisimple group, the character lattice of $\underline{G}$ coincides with the weight lattice. We say two characters λ_1 and $\lambda_2 \in X^*(\underline{S})$ are equivalent if they have the same image in $\frac{X^*(\underline{S})}{(p^r-\pi)X^*(\underline{S})}$. Let $\lambda = \sum n_\alpha\omega_\alpha \in X^*(\underline{S})$, where n_α are integers.

Claim λ is equivalent to $\lambda' = \sum n'_\alpha\omega_\alpha$ where all $n'_\alpha \geq 0$.

Proof. We have $\mu := (p^r-1)\sum\omega_\alpha = \sum(p^r-\pi)\omega_\alpha \in (p^r-\pi)X_*(\underline{S})$. For a sufficiently large integer N, $\lambda' = N\mu + \lambda$ is a dominant character. □

Now we assume λ is dominant. Among all dominant characters equivalent to λ, we assume $|\lambda| := \sum n_\alpha \in \mathbb{Z}_{\geq 0}$ is the smallest.

Next we show that $\lambda \in X_{p^r}(\underline{S})$. Assume some $n_\beta \geq p^r$. Write $n_\beta = s + p^r t$, $s, t \in \mathbb{Z}_{\geq 0}$. We have

$$\sum n_\alpha\omega_\alpha \sim \sum_{\alpha\neq\beta,\pi\beta} n_\alpha\omega_\alpha + s\omega_\beta + (n_{\pi\beta}+t)\omega_{\pi\beta} =: \lambda'.$$

We have

$$|\lambda| - |\lambda'| = n_\alpha - s - t = (p^r-1)t > 0$$

which contradicts the assumption that $|\lambda|$ is minimal among all equivalent dominant characters. □

6.1.5. Definition A based inertial Deligne-Lusztig datum $(\underline{S}, \underline{\chi}, \underline{B}_{\underline{S}})$ is said to be of *niveau* m if $\underline{B}_{\underline{S}}$ is $\mathbf{F}^m$-stable.

6.1.6. Proposition There exists a natural surjective map from the set of equivalence classes of irreducible representations of $\underline{G}^{\mathbf{F}^m}$ to the set of geometric conjugacy classes of niveau-m based inertial Deligne-Lusztig data. This map does not depend on any choices.

Proof. An irreducible representation of $\underline{G}^{\mathbf{F}^m}$ is the restriction of an irreducible algebraic representation of highest weight $\lambda \in X_{rm}(\underline{T})$ where $(\underline{T}, \underline{B})$ is a $\mathbf{F}^m$-stable Borel pair. The proposition follows from Lemma 6.1.4. □

6.2. Deligne-Lusztig representations

$\underline{G}$ is a connected reductive group over $\bar{\mathbb{F}}_p$ with simply-connected derived subgroup, equipped with a $\mathbb{F}_p$-structure $\mathbf{F} : \underline{G} \to \underline{G}$. Fix a $\mathbf{F}$-stable pinning $(\underline{B}, \underline{T}, \{u_\alpha\})$ of $\underline{G}$, which exists because all reductive groups over finite fields are quasi-split.

6.2.1. Herzig's presentation of Deligne-Lusztig datum In [Her09], a tame type is described by a pair $(w, \mu) \in \Omega(\underline{G}, \underline{T}) \times X^*(\underline{T})$.

Let $(\underline{S}, \underline{\chi}, \underline{B}_{\underline{S}})$ be a based inertial Deligne-Lusztig datum. There exists an element $g \in \underline{G}$ such that $(\underline{B}, \underline{T}) = g(\underline{B}_{\underline{S}}, \underline{S})g^{-1}$. Two different choices of g differ by left translation by an element of $\underline{T}$, so we get a well-defined identification of $X^*(\underline{T}) \cong X^*(\underline{S})$. By Equation (3), $\underline{\chi} \in X^*(\underline{S})/(p-\pi)X^*(\underline{S})$, and we let μ be the element of $X^*(\underline{T})$ lifting $\underline{\chi}$. Let $\Delta := \Delta(\underline{B}, \underline{T}) \cong \Delta(\underline{B}_{\underline{S}}, \underline{S})$ be the set of simple roots. Since $\underline{S}$ is $\mathbf{F}$-stable, $\mathbf{F}$ acts on $X^*(\underline{S})$; however, since $\underline{B}_{\underline{S}}$ is not $\mathbf{F}$-stable in general, $\Delta(\underline{B}_{\underline{S}}, \underline{S})$ is not $\mathbf{F}$-stable in general. There exists an element $w \in \Omega(\underline{G}, \underline{S}) \cong \Omega(\underline{G}, \underline{T})$ such that $\mathbf{F}(\Delta(\underline{B}_{\underline{S}}, \underline{S})) = w\Delta(\underline{B}_{\underline{S}}, \underline{S})$.

The based inertial Deligne-Lusztig datum $(\underline{S}, \underline{\chi}, \underline{B}_{\underline{S}})$ is presented by the pair (w, μ) by many authors.

6.2.2. Regular p-restricted weights An element $\lambda \in X_1(\underline{T})$ is said to be *regular* if $\langle \lambda, \alpha^\vee \rangle \in [0, p-1)$ for all $\alpha \in \Delta(\underline{B}, \underline{T})$.

6.2.3. Lemma The natural map in Proposition 6.1.6 is injective when restricted to irreducible representations of $\underline{G}^{\mathbf{F}}$ of regular p-restricted highest weight.

Proof. We keep notations in the proof of Proposition 6.1.6. Let $\lambda = n_\alpha \omega_\alpha$ and $\lambda' = n'_\alpha \omega'_\alpha$ such that $\lambda - \lambda' \in (p - \pi)X^*(\underline{T})$. Say

$$\lambda - \lambda' = \sum c_\alpha (p\omega_\alpha - \omega_{\pi\alpha}) = \sum (pc_\alpha - c_{\pi^{-1}\alpha})\omega_\alpha.$$

We have

$$-p < pc_\alpha - c_{\pi^{-1}\alpha} < p$$

for all $\alpha \in \Delta(\underline{B}, \underline{T})$. If c_α is the largest integer for various α, we have $p > pc_\alpha - c_{\pi^{-1}\alpha} \geq (p-1)c_\alpha$ and thus $c_\alpha \leq 1$. Similarly, $-p < (p-1)c_\alpha$ and thus $c_\alpha \in [-1, 1]$. We have $pc_\alpha - c_{\pi^{-1}\alpha} \in \{p-1, 0, -p+1\}$. In particular, if $n_\alpha \neq n'_\alpha$, then $\{n_\alpha, n'_\alpha\} = \{0, p-1\}$. □

6.2.4. Definition A based inertial Deligne-Lusztig datum of niveau 1 is said to be *regular* if it has a Herzig presentation (w, μ) where μ is a regular p-restricted weight.

An irreducible representation of $\underline{G}^{\mathbf{F}}$ is said to be *regular* if it is the restriction of a simple $\underline{G}$-module $L(\mu)$ to $\underline{G}^{\mathbf{F}}$ where μ is a regular p-restricted weight.

6.2.5. Proposition There is a natural bijection between equivalence classes of regular irreducible representations of $\underline{G}^{\mathbf{F}}$ and geometric conjugacy classes of regular based inertial Deligne-Lusztig data of niveau 1.

Proof. Combine Proposition 6.1.6 and Lemma 6.2.3. □

6.2.6. Mod p twisting element Following [GHS], we denote by $\eta \in X^*(\underline{T})$ an element that is Frobenius-stable and $\langle \eta, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta(\underline{B}, \underline{T})$.

6.2.7. Herzig's reflection operator $\mathcal{R}$ We define an involution operator on the set of isomorphism classes of regular irreducible representations of $\underline{G}^{\mathbf{F}}$ by

$$\mathcal{R} : L(\mu) \mapsto L(w_0 \cdot (\mu - p\eta))$$

where w_0 is the longest Weyl group element and η is a mod p twisting element.

6.2.8. Deligne-Lusztig induction Let $(\underline{S}, \underline{\chi})$ be an inertial Deligne-Lusztig datum. Write $\bar{V}(\underline{S}, \underline{\chi})$ for the reduction mod p of the Deligne-Lusztig induction $\varepsilon_{\underline{G}} \varepsilon_{\underline{S}} R_{\underline{S}}^{\underline{\chi}}$ (see [Her09, Section 4.1] for unfamiliar notations.)

6.3. A generalization of Herzig's recipe

Let G be a quasi-split tamely ramified reductive group over $\mathbb{Q}_p$. Fix a $\mathrm{Gal}_{\mathbb{Q}_p}$-stable pinning $(B, T, \{X_\alpha\})$ of G such that T is maximally unramified, which determines a superspecial parahoric $\mathcal{G}^\circ$ of G. Write $\underline{G}$ for the reductive quotient of $\mathcal{G}^\circ$. The pinning of G determines a Frobenius-stable pinning $(\underline{B}, \underline{T}, \{u_\alpha\})$ of $\underline{G}$.

6.3.1. Definition A *Serre weight* for G is an irreducible $\bar{\mathbb{F}}_p$-representation of $\mathcal{G}$, where $\mathcal{G}$ is the maximally bounded subgroup of G containing $\mathcal{G}^\circ$.

A *parahoric Serre weight* for G is an irreducible $\bar{\mathbb{F}}_p$-representation of $\underline{G}(\mathbb{F}_p)$.

6.3.2. Assumption Assume both G and $\underline{G}$ admit a local twisting element, and assume $\underline{G}$ has a simply-connected derived subgroup. More precisely, there exists an element $\eta_{\mathbb{Q}_p} \in X^*(T)^{\mathrm{Gal}_{\mathbb{Q}_p}}$ such that $\langle \eta_{\mathbb{Q}_p}, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta(B, T)$; and there exists an element $\eta_{\mathbb{F}_p} \in X^*(\underline{T})^{\mathrm{Gal}_{\mathbb{F}_p}}$ such that $\langle \eta_{\mathbb{F}_p}, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta(\underline{B}, \underline{T})$.

Write $T_0 \subset T$ for the maximally unramified subtorus. The identification $X^*(T_0) \cong X^*(\underline{T})$ allows us to define a reduction map

$$X^*(T) \xrightarrow{\text{restriction}} X^*(T_0) \to X^*(\underline{T}).$$

By abuse of notation, we denote by $\eta_{\mathbb{Q}_p}$ the image of $\eta_{\mathbb{Q}_p}$ in $X^*(\underline{T})$.

6.3.3. The speculative recipe Let $\tau : I_{\mathbb{Q}_p} \to {}^L G(\bar{\mathbb{F}}_p)$ be a tame inertial L-parameter. Define

$$W^?(\tau) := \mathcal{R}(\mathrm{JH}(\bar{V}(\mathrm{DL}^{-1}(\tau)) \otimes W(w_0(\eta_{\mathbb{F}_p} - \eta_{\mathbb{Q}_p})))).$$

Here DL is the Deligne-Lusztig map (see Section 4), w_0 is the longest Weyl group element, $W(-)$ denotes the Weyl module, and $\mathrm{JH}(-)$ denotes the set of Jordan-Hölder factors.

A. Maximal tori of quasi-split groups

In this appendix, we clarify the relation between maximal tori of a quasi-split group and that of its Langlands dual group in the natural generality.

Let F be a perfect field with a fixed separable closure F^s, and let G be a quasi-split group over F. Let k be an arbitrary algebraically closed field, and write ${}^L G$ for the Langlands dual group of G defined over k.

We start with recalling the classification of tori.

A.0.1. Proposition The set of F-isomorphism classes of F-tori of dimension n is in natural bijection with group homomorphisms $\mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}((\mathbb{G}_m^{\oplus n})_k)$ having finite image, up to $\mathrm{GL}_n(\mathbb{Z})$-conjugacy.

Proof. By [Crd11, Lemma 7.1.1], the F-isomorphism classes of F-tori are classified by the cohomology set $H^1(\mathrm{Gal}_F, \mathrm{Aut}(\mathbb{G}_m^{\oplus n})) = H^1(\mathrm{Gal}_F, \mathrm{GL}_n(\mathbb{Z}))$. The proposition follows from the fact that the Galois action on $\mathrm{GL}_n(\mathbb{Z})$ is trivial. □

A.0.2. Definition A *framed maximal torus* of G is a torus defined over F of dimension equal to the rank of G, together with an embedding $j : S \to G$ defined over F^s.

The Galois group Gal_F acts on the collection of framed maximal tori by $j \mapsto \sigma \circ j \circ \sigma^{-1} =: j^\sigma$, $\sigma \in \mathrm{Gal}_F$.

A *conjugacy class of framed maximal tori* is a set $\{g \circ j \circ g^{-1} =: \mathrm{Int}(g) \circ j | g \in G(F^s)\}$ where j is a framed maximal tori.

A.0.3. Theorem (Kottwitz) Let J be a conjugacy class of framed maximal tori which is stable under Gal_F-action. Then there exists some $j \in J$ which is an algebraic group homomorphism defined over F.

Proof. It is [Kot82, Corollary 2.2]. □

A.0.4. Definition On the dual side, we define a *framed maximal torus* of ${}^L G$ to be an k-torus $\widehat{S}$ of dimension $\mathrm{rk}\, G$ together with

- a Galois action $\psi : \mathrm{Gal}_F \to \mathrm{Aut}(\widehat{S})$ and
- an embedding $\hat{\jmath} : \widehat{S} \to \widehat{G}$.

Note that the embedding $\hat{\jmath}$ is arbitrary and does not have to respect Galois actions. The Galois group Gal_F acts on the collection of framed maximal tori by $(\psi, \hat{\jmath}) \mapsto (\psi, \sigma \circ \hat{\jmath} \circ \psi(\sigma)^{-1})$, $\sigma \in \mathrm{Gal}_F$.

A.0.5. Proposition There is a natural one-to-one correspondence $(S, j) \mapsto (\psi_S, \hat{\jmath})$ between Gal_F-stable conjugacy classes of framed maximal tori of G, and Gal_F-stable conjugacy classes of framed maximal tori of ${}^L G$ where the map $S \mapsto \psi_S$ is the dual map of the map defined in Proposition A.0.1.

Proof. See [Kal19a, Section 5.1] for the construction. Note that loc. cit. requires that $\mathrm{char}\, k = 0$, which is not necessary because the equivalence is validated by a Galois cohomology computation with $\Omega(T, G)$-coefficients, where $\Omega(T, G)$ is the absolute Weyl group and does not depend on k. □

References

[BC08] O. Brinon and B. Conrad. *CMI summer school notes on p-adic Hodge theory.*

[BG14] K. Buzzard and T. Gee. "The conjectural connections between automorphic representations and Galois representations". In: Oct. 2014, pp. 135–187. ISBN: 9781107691926. DOI: 10.1017/CBO9781107446335.006.

[BG19] R. Bellovin and T. Gee. "G-valued local deformation rings and global lifts". In: *Algebra and Number Theory* 13 (2019), pp. 333–378.

[Ch20] C. Birkbeck. "On the p-adic Langlands correspondence for algebraic tori". In: *J. Théor. Nombres Bordeaux* 32.1 (2020), pp. 133–158. ISSN: 1246-7405. URL: http://jtnb.cedram.org/item?id=JTNB_2020__32_1_133_0.

[Bo79] A. Borel. "Automorphic L-functions". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[BS53] A. Borel and J.-P. Serre. "Sur certains sous-groupes des groupes de Lie compacts." In: *Commentarii mathematici Helvetici* 27 (1953), pp. 128–139. URL: http://eudml.org/doc/139058.

[BT65] A. Borel and J. Tits. "Groupes réductifs". In: *Inst. Hautes Études Sci. Publ. Math.* 27 (1965), pp. 55–150. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1965__27__55_0.

[BT71] A. Borel and J. Tits. "Éléments unipotent et sous-groupes paraboliques de groupes réductifs. I". In: *Inventiones mathematicae* 12 (1971), pp. 95–104.

[Ca93] R. W. Carter. "Finite groups of lie type: conjugacy classes and complex characters". In: (1993).

[CGP15] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups.* Second. Vol. 26. New Mathematical Monographs. Cambridge University Press, Cambridge, 2015, pp. xxiv+665. ISBN: 978-1-107-08723-1. DOI: 10.1017/CBO9781316092439. URL: https://doi.org/10.1017/CBO9781316092439.

[CL11] X. Caruso and T. Liu. "Some bounds for ramification of p^n-torsion semi-stable representations". In: *J. Algebra* 325 (2011), pp. 70–96. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2010.10.005. URL: https://doi.org/10.1016/j.jalgebra.2010.10.005.

[C11] B. Conrad. "Lifting global representations with local properties". In: *Preprint* ().

[Crd11] B. Conrad. *Reductive group scheme (SGA3 Summer School 2011).*

[DeB06] S. DeBacker. "Parameterizing conjugacy classes of maximal unramified tori via Bruhat-Tits theory". In: *Michigan Math. J.* 54.1 (2006), pp. 157–178. ISSN: 0026-2285. DOI: 10.1307/mmj/1144437442. URL: https://doi.org/10.1307/mmj/1144437442.

[DL76] P. Deligne and G. Lusztig. "Representations of reductive groups over finite fields". In: *Ann. of Math. (2)* 103.1 (1976), pp. 103–161. ISSN: 0003-486X. DOI: 10.2307/1971021. URL: https://doi.org/10.2307/1971021.

[DR09] S. DeBacker and M. Reeder. "Depth-zero supercuspidal L-packets and their stability". In: *Ann. of Math. (2)* 169.3 (2009), pp. 795–901. ISSN: 0003-486X. DOI: 10.4007/annals.2009.169.795. URL: https://doi.org/10.4007/annals.2009.169.795.

[EG23] M. Emerton and T. Gee. "Moduli stacks of étale $(\phi, gamma)$-modules, and the existence of crystalline lifts". In: (2023).

[FLH] T. Feng and B. V. L. Hung. “Mirror symmetry and the Breuil-Mézard conjecture”. In: *In preparation* (2023).

[NCS18] N. Fakhruddin, C. Khare, and S. Patrikis. “Lifting irreducible Galois representations”. In: (2018). arXiv: 1810.05803 [math.NT].

[GHLS] T. Gee et al. “Potentially crystalline lifts of certain prescribed types”. In: *Documenta* 22 (2017), pp. 391–422.

[GHS] T. Gee, F. Herzig, and D. Savitt. “General Serre weight conjectures”. In: *Journal of the European Mathematical Society* (2018).

[GK14] T. Gee and M. Kisin. “The Breuil-Mézard conjecture for potentially Barsotti-Tate representations”. In: *Forum Math. Pi* 2 (2014), e1, 56. ISSN: 2050-5086. DOI: 10.1017/fmp.2014.1. URL: https://doi.org/10.1017/fmp.2014.1.

[GR10] B. H. Gross and M. Reeder. “Arithmetic invariants of discrete Langlands parameters”. In: *Duke Math. J.* 154.3 (2010), pp. 431–508. ISSN: 0012-7094. DOI: 10.1215/00127094-2010-043. URL: https://doi.org/10.1215/00127094-2010-043.

[Ha15] T. J. Haines. “On Satake parameters for representations with parahoric fixed vectors”. In: *Int. Math. Res. Not. IMRN* 20 (2015), pp. 10367–10398. ISSN: 1073-7928. DOI: 10.1093/imrn/rnu254. URL: https://doi.org/10.1093/imrn/rnu254.

[Ha18] T. J. Haines. “Dualities for root systems with automorphisms and applications to non-split groups”. In: *Represent. Theory* 22 (2018), pp. 1–26. DOI: 10.1090/ert/512. URL: https://doi.org/10.1090/ert/512.

[Her09] F. Herzig. “The weight in a Serre-type conjecture for tame n-dimensional Galois representations”. In: *Duke Math. J.* 149.1 (2009), pp. 37–116. ISSN: 0012-7094. DOI: 10.1215/00127094-2009-036. URL: https://doi.org/10.1215/00127094-2009-036.

[IM65] N. Iwahori and H. Matsumoto. “On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups”. In: *Publications mathématiques de l'I.H.E.S.* (1965), pp. 5–48.

[Iw86] K. Iwasawa. *Local class field theory*. Oxford Science Publications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1986, pp. viii+155. ISBN: 0-19-504030-9.

[Jan03] J. C. Jantzen. *Representations of algebraic groups*. Second. Vol. 107. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xiv+576. ISBN: 0-8218-3527-0.

[Kal19b] T. Kaletha. “Supercuspidal L-packets”. In: ().

[Kal19a] T. Kaletha. “Regular supercuspidal representations”. In: *J. Amer. Math. Soc.* 32.4 (2019), pp. 1071–1170. ISSN: 0894-0347. DOI: 10.1090/jams/925. URL: https://doi.org/10.1090/jams/925.

[Kal21] T. Kaletha. *On L-embeddings and double covers of tori over local fields*. 2021. arXiv: 1907.05173 [math.RT].

[KM22] K. Kozioł and S. Morra. “Serre weight conjectures for p-adic unitary groups of rank 2”. In: *Algebra Number Theory* 16.9 (2022), pp. 2005–2097. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2022.16.2005. URL: https://doi.org/10.2140/ant.2022.16.2005.

[Kot82] R. E. Kottwitz. “Rational conjugacy classes in reductive groups”. In: *Duke Math. J.* 49.4 (1982), pp. 785–806. ISSN: 0012-7094. URL: http://projecteuclid.org/euclid.dmj/1077315531.

[KP22] T. Kelatha and G. Prasad. *Bruhat-Tits Theory: A New Approach*. Cambridge University Press; New edition, 2022.

[KS12] R. Kottwitz and D. Shelstad. *On Splitting Invariants and Sign Conventions in Endoscopic Transfer*. 2012. DOI: 10.48550/ARXIV.1201.5658. URL: https://arxiv.org/abs/1201.5658.

[KS99] R. E. Kottwitz and D. Shelstad. "Foundations of twisted endoscopy". In: *Astérisque* 255 (1999), pp. vi+190. ISSN: 0303-1179.

[LLHLM22] D. Le et al. *Extremal weights and a tameness criterion for mod p Galois representations*. 2022. arXiv: 2206.06442 [math.NT].

[LLHLM23] D. Le et al. "Local models for Galois deformation rings and applications". In: *Invent. Math.* 231.3 (2023), pp. 1277–1488. ISSN: 0020-9910. DOI: 10.1007/s00222-022-01163-4. URL: https://doi.org/10.1007/s00222-022-01163-4.

[Lee23] H. Lee. *Emerton–Gee stacks, Serre weights, and Breuil–Mézard conjectures for* GSp_4. 2023. arXiv: 2304.13879 [math.NT].

[L22] Z. Lin. "Crystalline lifts and a variant of the Steinberg-Winter theorem". In: *Doc. Math.* 27 (2022), pp. 2441–2468. ISSN: 1431-0635,1431-0643.

[L23C] Z. Lin. *Heisenberg varieties and the existence of de Rham lifts*. 2023. arXiv: 2309.00761 [math.NT].

[L23] Z. Lin. *The Emerton-Gee stacks for tame groups, I*. 2023. arXiv: 2304.05317 [math.NT].

[L23B] Z. Lin. *The Emerton-Gee stacks for tame groups, II*. 2023. arXiv: 2309.05773 [math.NT].

[LS87] R. P. Langlands and D. Shelstad. "On the definition of transfer factors". In: *Math. Ann.* 278.1-4 (1987), pp. 219–271. ISSN: 0025-5831. DOI: 10.1007/BF01458070. URL: https://doi.org/10.1007/BF01458070.

[M18] J. Milne. *Reductive groups*. URL: https://www.jmilne.org/math/CourseNotes/RG.pdf.

[MS03] G. McNinch and E Sommers. "Component groups of unipotent centralizers in good characteristic". In: *Journal of Algebra* (2003), pp. 323–337.

[Mu13] A. Muller. "Relèvements cristallins de représentations galoisiennes". In: *Université de Strasbourg* (2013).

[Se87] J.-P. Serre. "Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$". In: *Duke Math. J.* 54.1 (1987), pp. 179–230. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-87-05413-5. URL: https://doi.org/10.1215/S0012-7094-87-05413-5.

[Se89] J.-P. Serre. *Abelian l-adic representations and elliptic curves*. Second. Advanced Book Classics. With the collaboration of Willem Kuyk and John Labute. Addison-Wesley Publishing Company, Redwood City, CA, 1989, pp. xxiv+184. ISBN: 0-201-09384-7.

[Se98] J.-P. Serre. "Moursund Lectures 1998". In: *arXiv* (1998). URL: https://arxiv.org/abs/math/0305257.

[So98] E Sommers. "A generalization of the Bala-Carter theorem". In: *IMRN* (1998), pp. 539–562.

[Sp98] T. A. Springer. *Linear algebraic groups*. Second. Boston: Birkhäuser Basel, 1998.

[St68] R. Steinberg. "Endomorphism of linear algebraic groups". In: *Memoirs of the AMS* (1968).

[Ti66] J. Tits. "Normalisateurs de tores. I. Groupes de Coxeter étendus". In: *J. Algebra* 4 (1966), pp. 96–116. ISSN: 0021-8693. DOI: 10.1016/0021-8693(66)90053-6. URL: https://doi.org/10.1016/0021-8693(66)90053-6.

[Yu09] J.-K. Yu. "On the local Langlands correspondence for tori". In: *Ottawa lectures on admissible representations of reductive p-adic groups*. Vol. 26. Fields Inst. Monogr. Amer. Math. Soc., Providence, RI, 2009, pp. 177–183. DOI: 10.1090/fim/026/07. URL: https://doi.org/10.1090/fim/026/07.

[Zhu15] X. Zhu. "The geometric Satake correspondence for ramified groups". In: *Ann. Sci. Éc. Norm. Supér. (4)* 48.2 (2015), pp. 409–451. ISSN: 0012-9593. DOI: 10.24033/asens.2248. URL: https://doi.org/10.24033/asens.2248.

[Zhu21] X. Zhu. "A note on Integral Satake isomorphisms". In: (May 2020).