THE EMERTON-GEE STACKS FOR TAME GROUPS, II

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Abstract. We construct the moduli stacks of potentially semistable L-parameters for tame p-adic groups. As an application, we give a recursive description of the irreducible components of the reduced Emerton-Gee stacks for classical groups. Our approach is grounded in a formalized framework whose axioms were previously confirmed in a separate preprint for both ramified and unramified unitary groups. When specialized to even unitary groups, we show the irreducible components of the reduced Emerton-Gee stacks are in natural bijection with parahoric Serre weights.

Contents

1. Introduction

Let G be a quasi-split reductive group over K that splits over E, where E is a tamely ramified extension of $\mathbb{Q}_p$. Write ${}^LG = \widehat{G} \rtimes \mathrm{Gal}(E/K)$ for the Langlands dual group of G, where $\widehat{G}$ is the pinned dual group of G defined over $\operatorname{Spec} \mathbb{Z}$.

In this paper, we continue the discussion of [L23B]. We first generalize the construction of potentially semistable moduli stacks to general groups.

Theorem 1. *(Theorem 2.6.3) Let τ be an inertial type and let $\underline{\lambda}$ be a Hodge type. There exists a p-adic formal algebraic stack $\mathcal{X}^{\mathrm{ss},\tau,\underline{\lambda}}_{K,{}^LG}$ of finite type over $\operatorname{Spf}\mathcal{O}$, which is uniquely determined as the $\mathcal{O}$-flat closed substack of $\mathcal{X}_{K,{}^LG}$ by the following property: if A° is a finite $\mathcal{O}$-flat algebra, then $\mathcal{X}^{\mathrm{ss},\tau,\underline{\lambda}}_{K,{}^LG}(A^\circ)$ is the subgroupoid consisting of L-parameters which become potentially semistable of Hodge type $\underline{\lambda}$ and inertia type τ after inverting p.*

The mod p fiber

$$\mathcal{X}^{\mathrm{ss},\tau,\underline{\lambda}}_{K,{}^LG} \underset{\operatorname{Spf}\mathcal{O}}{\times} \operatorname{Spec}\mathbb{F}$$

is equidimensional of dimension $\dim_{\mathbb{Q}_p} \widehat{G}/P_{\underline{\lambda}}$.

Theorem 1 combined with the techniques introduced in [L23C] allows us to classify irreducible components of the reduced Emerton-Gee stacks $\mathcal{X}_{K,{}^LG,\mathrm{red}}$; see Theorem 2 below. The inputs for Theorem 2 are, roughly speaking, the main theorems of [L23C], but axiomized and formalized for reductive groups of type A, B, C, or D under the terminology *groups admitting a classical structure* (Definition 4.2.3).

Under the setup of Definition 4.2.3, $\widehat{G}$ admits a distinguished parabolic $\widehat{P}$ that is $\mathrm{Gal}(E/K)$-stable, which we call the niveau 1 maximal proper parabolic. Write ${}^LP$ for $\widehat{P} \rtimes \mathrm{Gal}(E/K) \subset {}^LG$. The Levi factor ${}^LM$ of ${}^LP$ is the L-group of a reductive group $M \cong \operatorname{Res}_{E/K}\mathrm{G}_m \times H_M$ for some reductive group H_M. Let $\widehat{B}$ be a $\mathrm{Gal}(E/K)$-stable Borel of $\widehat{G}$, and write ${}^LB$ for $\widehat{B} \rtimes \mathrm{Gal}(E/K)$. We have studied the parabolic versions of the Emerton-Gee stacks $\mathcal{X}_{K,{}^LP}$ in [L23B, Section 10]. In Subsection 4.4, we investigate the *maximally non-split* part $\coprod_i \mathcal{X}^{\mathrm{mns}}_{K,{}^LB,i,\mathrm{red}}$ of $\mathcal{X}_{K,{}^LB}$, and conclude

Lemma 1. *(Lemma 4.5.6) The morphism*

$$\coprod_i \mathcal{X}^{\mathrm{mns}}_{K,{}^LB,i,\mathrm{red}} \to \mathcal{X}_{K,{}^LG,\mathrm{red}}$$

induces a bijection between irreducible components of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{K,{}^LB,i,\mathrm{red}}$ of maximal dimension and irreducible components of $\mathcal{X}_{K,{}^LG,\mathrm{red}}$.

The group homomorphisms

$$\begin{tikzcd} {}^LB \arrow[r] & {}^LP \arrow[r] \arrow[d] & {}^LG \\ & {}^LM & \end{tikzcd}$$

induce morphisms of stacks

$$\begin{tikzcd} \coprod_i \mathcal{X}^{\mathrm{mns}}_{K,{}^LB,i,\mathrm{red}} \arrow[r] & \mathcal{X}_{K,{}^LP} \arrow[r] \arrow[d] & \mathcal{X}_{K,{}^LG} \\ & \mathcal{X}_{K,{}^LM} & \end{tikzcd}$$

By Lemma 1, irreducible components of $\mathcal{X}_{K,{}^LG,\mathrm{red}}$ are identified with irreducible components of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{K,{}^LB,i,\mathrm{red}}$ of maximal dimension. We say an irreducible component of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{K,{}^LB,i,\mathrm{red}}$ (or $\mathcal{X}_{K,{}^LG,\mathrm{red}}$) of maximal dimension is *relatively Steinberg* if its scheme-theoretic image in $\mathcal{X}_{K,{}^LM,\mathrm{red}}$ is *not* an irreducible component of $\mathcal{X}_{K,{}^LM,\mathrm{red}}$; and we say it is *relatively non-Steinberg* if otherwise.

Theorem 2. *(Theorem 4.5.8) Let U be the unipotent radical of ${}^LP$. Assume either*

- *H_M is not a torus or*
- *there exists a surjection $\mathrm{Res}_{E/K}\,\mathrm{G}_m \to H_M$ (for example, if $H_M \cong \mathrm{Res}_{E/K}\,\mathrm{G}_m$ or U_1).*

Then the following are true.

(1) If $\dim U/[U,U] \geq 2$, then there exists a natural bijection between the irreducible components of $\mathcal{X}_{K,{}^LH_M,\mathrm{red}}$ and the relatively Steinberg irreducible components of $\mathcal{X}_{K,{}^LG,\mathrm{red}}$.

(2) There exists a natural bijection between the irreducible components of $\mathcal{X}_{K,{}^LM,\mathrm{red}}$ and the relatively non-Steinberg irreducible components of $\mathcal{X}_{K,{}^LG,\mathrm{red}}$.

Write U_n for the quasi-split unitary group over K which splits over a quadratic extension E/K. By the main results of [L23C], we have the following:

Theorem 3. *(Theorem 5.0.2) There exists a bijection between the irreducible components of $\mathcal{X}_{K,{}^LU_n,\mathrm{red}}$ and the irreducible components of $\mathcal{X}_{K,{}^L(\mathrm{Res}_{E/K}\mathrm{G}_m\times U_{n-2}),\mathrm{red}} \amalg \mathcal{X}_{K,{}^LU_{n-2},\mathrm{red}}$.*

In the even unitary case, the reductive quotient of the superspecial parahoric have simply-connected derived subgroup. As a consequence, we have the following:

Theorem 4. *(Corollary 5.0.3) Assume $K = \mathbb{Q}_p$. There exists a bijection between the irreducible components of $\mathcal{X}_{K,{}^LU_{2m},\mathrm{red}}$ and the parahoric Serre weights for U_{2m}.*

1.0.1. Notations We freely use the moduli stacks constructed in [L23B], see [L23B, Table 1.7.1].

We will use dynamic methods to study parabolic subgroups, see [L23, Section 2.1.1].

In Section 2, G is a connected reductive group over a p-adic field K and split over E.

In Section 4 and Section 5, G is a connected reductive group over a p-adic field F and splits over K.

2. Moduli of potentially semistable L-parameters

2.1. Breuil-Kisin modules with $\widehat{G}$-structure

Let $\mathcal{O} \supset \mathcal{O}_K$ be a DVR over $\mathbb{Z}_p$.

Denote by k the residue field of K. Let A be a $\mathbb{Z}_p$-algebra. For each choice of a compatible family $\pi^{1/p^\infty} = (\pi^{1/p^n})_{n\in\mathbb{Z}_+}$ of p-power roots of a uniformizer of K in $\bar{\mathbb{Q}}_p$, we define an embedding

$$(W(k)\otimes_{\mathbb{Z}_p} A)[[u]] \to \mathbb{A}_{\mathrm{inf},A}$$
$$u \mapsto [\pi^\flat]$$

where $\pi^\flat = \varprojlim_n \pi^{1/p^n} \in \mathcal{O}_{\mathbb{C}}^\flat$. Denote by $\mathfrak{S}_{\pi^\flat,A}$ the image of the embedding above.

2.1.1. Definition A projective *Breuil-Kisin module* with A-coefficients is a finitely generated projective $\mathfrak{S}_{\pi^\flat,A}$-module $\mathfrak{M}$, equipped with a φ-semi-linear morphism $\phi_{\mathfrak{M}}: \mathfrak{M}\to\mathfrak{M}$ such that $1\otimes\phi_{\mathfrak{M}}: \varphi^*\mathfrak{M}[1/E_{\pi^\flat}] \to \mathfrak{M}[1/E_{\pi^\flat}]$ is a bijection. Here $E_{\pi^\flat}$ is the Eisenstein polynomial corresponding to π.

We say $(\mathfrak{M},\phi_{\mathfrak{M}})$ is an *effective* Breuil-Kisin module if $\phi_{\mathfrak{M}}(\mathfrak{M})\subset\mathfrak{M}$. We say a Breuil-Kisin module $(\mathfrak{M},\phi_{\mathfrak{M}})$ has *height h* if

$$E_{\pi^\flat}^h\mathfrak{M} \subset \mathrm{Im}(1\otimes\mathfrak{M}) \subset \frac{1}{E_{\pi^\flat}^h}\mathfrak{M}.$$

Following the notation of [EG23], we denote by $\mathcal{C}_{\pi^\flat,d,h}$ the (limit-preserving) moduli stack of effective Breuil-Kisin modules of height at most h for the uniformizer π (see [EG23, 4.5.7] for the definition). Also denote by $\mathcal{C}^a_{\pi^\flat,d,h}$ the base-changed stack $\mathcal{C}_{\pi^\flat,d,h} \otimes_{\mathbb{Z}_p} \mathcal{O}/\varpi^a$.

Denote by $\mathcal{R}_{\pi^\flat,d}$ the moduli of rank-d étale φ-modules for the φ-ring $\mathfrak{S}_{\pi^\flat}[1/u]$, and denote by $\mathcal{R}^a_{\pi^\flat,d}$ the base-changed version of $\mathcal{R}_{\pi^\flat,d}$.

In [L23B, Section 3], we constructed the moduli stack $\mathcal{C}_{\pi^\flat,\widehat{G},h}$ of Breuil-Kisin modules with $\widehat{G}$-structure, and the moduli stack $\mathcal{R}_{\pi^\flat,\widehat{G}}$ of rank-d étale φ-modules with $\widehat{G}$-structure. Note that when $\widehat{G} = \mathrm{GL}_d$, there is a canonical monomorphism

$$\mathcal{C}_{\pi^\flat,d,h} \hookrightarrow \mathcal{C}_{\pi^\flat,\mathrm{GL}_d,h}$$

as $\mathcal{C}_{\pi^\flat,\mathrm{GL}_d,h}$ classifies Breuil-Kisin modules of height h that are not necessarily effective. Moreover, the h-th Tate twist morphism

$$\mathcal{C}_{\pi^\flat,\mathrm{GL}_d,h} \xrightarrow{\cong} \mathcal{C}_{\pi^\flat,d,2h}$$

sending F to $F \times^{\mathrm{GL}_d} \mathrm{G}_a^{\oplus d}(h)$ is an equivalence of 2-categories.

2.1.2. Canonical extensions of Gal_{K_∞}-actions Write K_s for $K(\pi^{1/p^s})$, $0 \le s \le \infty$. We also write $K_{\pi^\flat,s} = K_s$ to emphasize the choice of $\pi^\flat$.

By Fontaine's theory ([EG23, Proposition 2.7.8]), we can attach to a Breuil-Kisin module a $(\varphi, \mathrm{Gal}_{K_\infty})$-module. By [EG23, Proposition 4.5.8], the Gal_{K_∞}-action on a $(\varphi, \mathrm{Gal}_{K_\infty})$-module admits a canonical extension to Gal_{K_s}, where s is any integer greater than the constant $s(a, h, N)$ defined in [EG23, Lemma 4.3.3] (where $N > \frac{e(a+h)}{p-a}$ is a fixed number). More precisely, there exists a commutative diagram

(1)
$$\begin{tikzcd} & \mathcal{C}^a_{\pi^\flat,d,h} \arrow[dl] \arrow[d] \\ \mathcal{X}^a_{K_s,d} \arrow[r] & \mathcal{R}^a_{\pi^\flat,d} \end{tikzcd}$$

where $\mathcal{X}^a_{K_s,d} = \mathcal{X}_{K_s,d} \otimes_{\mathbb{Z}_p} \mathcal{O}/\varpi^a$ is the base-changed Emerton-Gee stack.

2.1.3. Proposition For each $s > s(a, 2h, N)$, there exists a commutative diagram

$$\begin{tikzcd} & \mathcal{C}^a_{\pi^\flat,d,h} \arrow[d, hook] \\ & \mathcal{C}^a_{\pi^\flat,\mathrm{GL}_d,h} \arrow[dl] \arrow[d] \\ \mathcal{X}^a_{K_s,d} \arrow[r] & \mathcal{R}^a_{\pi^\flat,d} \end{tikzcd}$$

extending Diagram (1).

Proof. It follows immediately from the discussion before Paragraph 2.1.2. □

2.1.4. Definition Fix once for all an embedding $\widehat{G} \hookrightarrow \mathrm{GL}_d$. We say a Breuil-Kisin module (F, ϕ_F) with $\widehat{G}$-structure has *height* h if $(F, \phi_F) \times^{\widehat{G}} \mathrm{GL}_d$ has height h.

2.1.5. Lemma For each $s \geq 0$, the morphism

$$(\dagger) \qquad \mathcal{X}^a_{K_s,\widehat{G}} \hookrightarrow \mathcal{X}^a_{K_s,\mathrm{GL}_d} \underset{\mathcal{R}^a_{\pi^\flat,\mathrm{GL}_d}}{\times} \mathcal{R}^a_{\pi^\flat,\widehat{G}}$$

is a closed immersion.

Proof. Write $V = \mathbb{G}_a^{\oplus d}$ for the trivial vector space of rank d so that $\mathrm{GL}_d = \mathrm{GL}(V)$. For ease of notation, set $s = 0$. Fix a finite type $\mathcal{O}/\varpi^a$-algebra A. $\mathcal{X}^a_{K,\mathrm{GL}_d}(A)$ is equivalent to the groupoid of projective étale $(\varphi, \mathrm{Gal}_K)$-modules of rank d with A-coefficients, and $\mathcal{R}^a_{\pi^\flat,\mathrm{GL}_d}(A)$ is equivalent to the groupoid of projective étale $(\varphi, \mathrm{Gal}_{K_\infty})$-modules of rank d with A-coefficients. An object of $\mathcal{X}^a_{K,\mathrm{GL}_d} \underset{\mathcal{R}^a_{\pi^\flat,\mathrm{GL}_d}}{\times} \mathcal{R}^a_{\pi^\flat,\widehat{G}}(A)$ is a tuple

$$(M, F, \iota) := ((M, \phi_M, \rho_M), (F, \phi_F, \rho_{F,\infty}), \iota : (M, \phi_M, \rho_M|_{\mathrm{Gal}_{K_\infty}}) \cong (F, \phi_F, \rho_{F,\infty}) \times^{\widehat{G}} V),$$

where $(M, \phi_M, \rho_M) \in \mathcal{X}^a_{K,\mathrm{GL}_d}(A)$ and $(F, \phi_F, \rho_{F,\infty}) \in \mathcal{R}^a_{\pi^\flat,\widehat{G}}(A)$. A morphism

$$(M_1, F_1, \iota_1) \to (M_2, F_2, \iota_2)$$

is a pair $(g : M_1 \to M_2, f : F_1 \to F_2)$ such that

$$g = \iota_2^{-1} \circ (f \times^{\widehat{G}} V) \circ \iota_1.$$

The morphism $(\dagger)$ can be explicitly written as

$$(F, \phi_F, \rho_F) \mapsto (F \times^{\widehat{G}} V, F, \mathrm{id}).$$

From the description above, the morphism $(\dagger)$ is clearly faithful; indeed $(\dagger)$ is fully faithful: a $\widehat{G}$-torsor morphism $F_1 \to F_2$ respects the Gal_K-action if and only if $F_1 \times^{\widehat{G}} \mathrm{GL}_d \to F_1 \times^{\widehat{G}} \mathrm{GL}_d$ respects the Gal_K-action since $F_i \hookrightarrow F_i \times^{\widehat{G}} \mathrm{GL}_d$ is a closed subscheme $(i = 1, 2)$.

By [L23B, Lemma 10.3.2], $(\dagger)$ is of strong Ind-finite type in the sense of [L23B, Definition 10.3.1]. To show $(\dagger)$ is a closed immersion, it suffices to show it is proper using the (Noetherian) valuative criterion. Let Λ be a discrete valuation ring over $\bar{\mathbb{F}}_p$ with fraction field Ω. By [Stacks, Tag 0ARL], it is harmless to assume Λ is a complete discrete valuation ring; thus $\mathfrak{S}_{\pi^\flat,\Omega}$ is a disjoint union of the spectrum of Noetherian complete regular local rings over $\bar{\mathbb{F}}_p$. By the Grothendieck-Serre conjecture (see the main theorem of [FP15]), all $\widehat{G}$-torsors over $\mathfrak{S}_{\pi^\flat,\Lambda}$ are trivial $\widehat{G}$-torsors. The valuative criterion can be checked by noticing the simple fact that $\widehat{G}(\mathfrak{S}_{\pi^\flat,\Omega}) \cap \mathrm{GL}(\mathfrak{S}_{\pi^\flat,\Lambda}) = \widehat{G}(\mathfrak{S}_{\pi^\flat,\Lambda})$. $\square$

2.1.6. Lemma The diagram

$$\begin{tikzcd}
\mathcal{X}^a_{K_s,\widehat{G}} \underset{\mathcal{X}^a_{K_s,\mathrm{GL}_d}}{\times} \mathcal{C}^a_{\pi^\flat,\mathrm{GL}_d,h} \arrow[r, hook] \arrow[d] & \mathcal{R}^a_{\pi^\flat,\widehat{G}} \underset{\mathcal{R}^a_{\pi^\flat,\mathrm{GL}_d}}{\times} \mathcal{C}^a_{\pi^\flat,\mathrm{GL}_d,h} \arrow[d] \\
\mathcal{X}^a_{K_s,\widehat{G}} \arrow[r, hook] & \mathcal{R}^a_{\pi^\flat,\widehat{G}} \underset{\mathcal{R}^a_{\pi^\flat,\mathrm{GL}_d}}{\times} \mathcal{X}^a_{K_s,\mathrm{GL}_d}
\end{tikzcd}$$

is Cartesian.

Proof. Note that $A \underset{B \underset{C}{\otimes} D}{\otimes} (B \underset{C}{\otimes} E) = A \underset{B \underset{C}{\otimes} D}{\otimes} (B \underset{C}{\otimes} D) \otimes_D E = A \otimes_D E$. $\square$

2.1.7. Canonical extensions of Gal_{K_∞}-actions, the $\widehat{G}$-version
By [L23B, Corollary 3.7.2], the morphism

$$\mathcal{C}_{\pi^\flat, \widehat{G}, h} \hookrightarrow \mathcal{C}_{\pi^\flat, \mathrm{GL}_d, h} \underset{\mathcal{R}_{\pi^\flat, \mathrm{GL}_d}}{\times} \mathcal{R}_{\pi^\flat, \widehat{G}}$$

is a closed immersion. Define the stack $\mathcal{C}^a_{\pi^\flat, K_s, \widehat{G}, h}$ so that the following diagram is Cartesian

(2)
$$\begin{tikzcd}
\mathcal{C}^a_{\pi^\flat, K_s, \widehat{G}, h} \arrow[r, hook] \arrow[d, hook] & \mathcal{C}^a_{\pi^\flat, \widehat{G}, h} \arrow[d, hook] \\
\mathcal{X}^a_{K_s, \widehat{G}} \underset{\mathcal{X}^a_{K_s, \mathrm{GL}_d}}{\times} \mathcal{C}^a_{\pi^\flat, \mathrm{GL}_d, h} \arrow[r, hook] & \mathcal{R}^a_{\pi^\flat, \widehat{G}} \underset{\mathcal{R}^a_{\pi^\flat, \mathrm{GL}_d}}{\times} \mathcal{C}^a_{\pi^\flat, \mathrm{GL}_d, h}
\end{tikzcd}$$

Note that all arrows in the diagram above are closed immersions. The diagram above can be interpreted as follows: for all Breuil-Kisin module with $\widehat{G}$-structure, the Gal_{K_∞}-action can be extended to a Gal_{K_s}-action canonically in a way not necessarily compatible with the $\widehat{G}$-structure; and the condition that the canonical extension is compatible with the $\widehat{G}$-structure is a closed condition.

Define the stack $\mathcal{C}_{\pi^\flat, s, \widehat{G}, h}$ so that the following diagram is Cartesian

$$\begin{tikzcd}
\mathcal{C}^a_{\pi^\flat, s, \widehat{G}, h} \arrow[r] \arrow[d] & \mathcal{C}^a_{\pi^\flat, K_s, \widehat{G}, h} \arrow[d] \\
\mathcal{X}^a_{K, \widehat{G}} \arrow[r] & \mathcal{X}^a_{K_s, \widehat{G}}
\end{tikzcd}$$

is Cartesian. The stack $\mathcal{C}^a_{\pi^\flat, s, \widehat{G}, h}$ can be interpreted as the moduli stack of Breuil-Kisin modules $(\mathfrak{M}, \phi_{\mathfrak{M}})$ together with an enhancement: a $(\varphi, \mathrm{Gal}_{K_s})$-module $(\mathfrak{M} \underset{\mathfrak{S}_{\pi^\flat, A}[1/u]}{\otimes} W(\mathbb{F}^\flat)_A, \phi_{\mathfrak{M}} \otimes 1, \rho)$ such that $\rho \times^{\widehat{G}} \mathrm{GL}_d|_{\mathrm{Gal}_{K_s}}$ is the canonical Gal_{K_s}-action.

2.1.8. Lemma (1) The morphism $\mathcal{C}^a_{\pi^\flat, K_s, \widehat{G}, h} \to \mathcal{X}^a_{K_s, \widehat{G}}$ is representable by algebraic spaces, proper, and of finite presentation.

(2) The diagonal of $\mathcal{C}^a_{\pi^\flat, s, \widehat{G}, h}$ is affine and of finite presentation.

Proof. (1) It follows from the diagram (2) and [EG23, Lemma 4.5.9].

(2) It follows from part (1), [L23B, Theorem 7.1.2], and [EG23, Lemma 4.5.14]. $\square$

2.2. Breuil-Kisin-Fargues modules with $\widehat{G}$-structure

Let $L/\mathbb{Q}_p$ be a finite Galois extension, Let $\mathcal{O} \supset \mathcal{O}_L$ be a DVR over $\mathbb{Z}_p$.

Denote by l the residue field of L. Let A be a $\mathbb{Z}_p$-algebra. For each choice of a compatible family $\pi^{1/p^\infty} = (\pi^{1/p^n})_{n\in\mathbb{Z}_+}$ of p-power roots of a uniformizer of L in $\bar{\mathbb{Q}}_p$, we define an embedding

$$(W(l) \otimes A)[[u]] \to \mathbb{A}_{\mathrm{inf},A}$$
$$u \mapsto [\pi^\flat]$$

where $\pi^\flat = \varprojlim_n \pi^{1/p^n} \in \mathcal{O}_{\mathbb{C}}^\flat$. Denote by $\mathfrak{S}_{\pi^\flat,A}$ the image of the embedding above.

2.2.1. Definition A projective *Breuil-Kisin-Fargues* module with A-coefficients is a finitely generated projective $\mathbb{A}_{\mathrm{inf},A}$-module $\mathfrak{M}^{\mathrm{inf}}$, equipped with a φ-semi-linear endomorphism $\phi_{\mathfrak{M}^{\mathrm{inf}}} : \mathfrak{M}^{\mathrm{inf}} \to \mathfrak{M}^{\mathrm{inf}}$ such that $1 \otimes \phi_{\mathfrak{M}^{\mathrm{inf}}} : \varphi^*\mathfrak{M}^{\mathrm{inf}}[1/\xi] \to \mathfrak{M}^{\mathrm{inf}}[1/\xi]$ is a bijection. Here ξ is a generator of $\ker(\theta : \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}})$ ([BMS18, Definition 4.22]).

We say a Breuil-Kisin-Fargues module $\mathfrak{M}^{\mathrm{inf}}$ *descends to* $\mathfrak{S}_{\pi^\flat,A}$ if there is a Breuil-Kisin module $\mathfrak{M}_{\pi^\flat} \subset (\mathfrak{M}^{\mathrm{inf}})^{\mathrm{Gal}_{K_{\pi^\flat,\infty}}}$ such that $\mathbb{A}_{\mathrm{inf},A} \underset{\mathfrak{S}_{\pi^\flat,A}}{\otimes} \mathfrak{M}_{\pi^\flat} = \mathfrak{M}^{\mathrm{inf}}$. We say $\mathfrak{M}^{\mathrm{inf}}$ *admits all descents over* L if it descends to $\mathfrak{S}_{\pi^\flat,A}$ for every choice of $\pi^\flat$ (for every choice of π) and if furthermore

1. the $W(l)\otimes_{\mathbb{Z}_p} A$-submodule $\mathfrak{M}_{\pi^\flat}/[\pi^\flat]\mathfrak{M}_{\pi^\flat}$ of $(W(\bar{l})\otimes A)\otimes_{\mathbb{A}_{\mathrm{inf},A}} \mathfrak{M}^{\mathrm{inf}}$ is independent of the choice of π and $\pi^\flat$;
2. the $\mathcal{O}_L\otimes_{\mathbb{Z}_p} A$-submodule $\varphi^*\mathfrak{M}_{\pi^\flat}/E_{\pi^\flat}\varphi^*\mathfrak{M}_{\pi^\flat}$ of $\mathcal{O}_{\mathbb{C},A}\otimes_{\mathbb{A}_{\mathrm{inf},A}} \varphi^*\mathfrak{M}^{\mathrm{inf}}$ is independent of the choice of π and $\pi^\flat$.

2.2.2. Lemma The category of Breuil-Kisin-Fargues modules with A-coefficients that admits all descents over L is an exact, rigid, symmetric monoidal category.

Proof. Note that the property of admitting all descents over L is preserved under tensor products and duals. □

2.2.3. Definition Let A be a p-adically complete $\mathcal{O}$-algebra which is topologically of finite type. A *Breuil-Kisin-Fargues* Gal_L*-module with* A*-coefficients* is a Breuil-Kisin-Fargues module $(\mathfrak{M}^{\mathrm{inf}}, \phi_{\mathfrak{M}^{\mathrm{inf}}})$ with A-coefficients equipped with a continuous semilinear Gal_L-action that commutes with $\phi_{\mathfrak{M}^{\mathrm{inf}}}$.

2.2.4. Connection to semistable Galois representations

Let M be an étale $(\varphi, \mathrm{Gal}_L)$-module. By [EG23, Section 2.7], we can attach a Gal_L-representation $V(M)$ to M. By [EG23, Theorem F.11], if $V(M)$ is semistable with Hodge-Tate weights in $[0, h]$, then there exists a unique Breuil-Kisin-Fargues Gal_L-module $\mathfrak{M}^{\mathrm{inf}}$ of height at most h that admits all descents over L such that $\mathfrak{M}^{\mathrm{inf}} \otimes_{\mathbb{A}_{\mathrm{inf}}} W(\mathbb{C}^\flat) = M$. Moreover, by [EG23, Proposition 4.4.1], there exists a constant $s'(L, a, h, N)$ (where $N > \frac{e(a+h)}{p-a}$ is a fixed constant), such that for any choice $\pi^\flat$ with corresponding descent $\mathfrak{M}_{\pi^\flat}$ and for any $s > s'(L, a, h, N)$, the restriction to $\mathrm{Gal}_{L_{\pi^\flat,s}}$ of the action of Gal_L on $\mathfrak{M}^{\mathrm{inf}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^a$ agrees with the canonical action considered in Paragraph 2.1.2.

2.2.5. Definition A *Breuil-Kisin-Fargues* Gal_L*-module with* A*-coefficients with* $\widehat{G}$*-structure* is a faithful, exact, symmetric monoidal functor from ${}^f\mathrm{Rep}_{\widehat{G}}$ to the category of Breuil-Kisin-Fargues Gal_L-module with A-coefficients.

2.2.6. Proposition Let F be an étale $(\varphi, \mathrm{Gal}_L)$-module with $\widehat{G}$-structure. Then $V(F)$ is a semistable $\widehat{G}$-valued Galois representation of Gal_L such that $V(F) \times^{\widehat{G}} \mathrm{GL}_d$ has Hodge-Tate weights in $[-h, h]$ if and only if there exists a (necessarily) unique Breuil-Kisin-Fargues Gal_L-module F^{inf} with $\widehat{G}$-structure which admits all descents over L and which satisfies $F = F^{\mathrm{inf}} \otimes_{\mathbb{A}_{\mathrm{inf}}} W(\mathbb{C}^\flat)$, and such that $F^{\mathrm{inf}} \times^{\widehat{G}} \mathrm{GL}_d$ is of height at most h.

Moreover, for any choice $\pi^\flat$ with corresponding descent $F_{\pi^\flat}$ and for any $s > s'(L, a, 2h, N)$, the restriction to $\mathrm{Gal}_{L_{\pi^\flat,s}}$ of the action of Gal_L on $(F^{\mathrm{inf}} \times^{\widehat{G}} \mathrm{GL}_d) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^a$ agrees with the canonical action considered in Paragraph 2.1.2.

Proof. Let $x \in^f \mathrm{Rep}_{\widehat{G}}$. By [Lev13, Proposition 5.3.2, Definition 5.3.1], $V(F) \times^{\widehat{G}} x = V(F \times^{\widehat{G}} x)$ is semistable. Fix a choice of $\pi^\flat$. By [EG23, Theorem F.11], $F \times^{\widehat{G}} x$ corresponds to a unique Breuil-Kisin-Fargues Gal_L-module F_x^{inf}. The uniqueness implies the association $F^{\mathrm{inf}} : x \mapsto F_x^{\mathrm{inf}}$ is functorial and monoidal. Since $F^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}) = F$ is faithful and exact and that A_{inf} is a subring of $W(\mathbb{C})$, F^{inf} is automatically faithful and lef-exact. It remains to show F^{inf} is right-exact, and we do it by the bundle extension technique.

By [EG23, Lemma 4.2.8], F_x^{inf} descends for each $\pi^\flat$ uniquely to a Breuil-Kisin module $F_x^{\pi^\flat}$. Since $\mathfrak{S}_{\pi^\flat} \to A_{\mathrm{inf}}$ is faithfully flat, it suffices to show the functor $F^{\pi^\flat} : x \mapsto F_x^{\pi^\flat}$ is right-exact. By [Lev13, Lemma 4.2.22], the functor $F^{\pi^\flat}[1/p]$ is exact, and thus defines a φ-module with $\widehat{G}$-structure F_1 over $\operatorname{Spec} \mathfrak{S}_{\pi^\flat}[1/p]$. The functor $F^{\pi^\flat}[1/u]$ is also exact by almost étale descent ($\mathfrak{S}[1/u] \to W(\mathbb{C})$ is faithfully flat, see [EG23, Proposition 2.2.14]), and thus defines a φ-module with $\widehat{G}$-structure F_2 over $\operatorname{Spec} \mathfrak{S}_{\pi^\flat}[1/u]$. The two φ-modules F_1 and F_2 can be glued along the intersection of $\operatorname{Spec} \mathfrak{S}_{\pi^\flat}[1/p]$ and $\operatorname{Spec} \mathfrak{S}_{\pi^\flat}[1/u]$. By [Lev13, Lemma 5.1.1] (see also the main result of [An22]), there exists unique extension F_3 of F_1 and F_2 to $\operatorname{Spec} \mathfrak{S}_{\pi^\flat}$. Since F_3, when regarded as a monoidal functor, is forced to be $F^{\pi^\flat}$, we have finished showing F^{inf} is a faithful, exact, symmetric monoidal functor.

The "moreover" part is [EG23, Proposition 4.4.1]. $\square$

The proposition above motivates the following definition.

2.2.7. Definition For each $h \geq 0$ and each finite Galois extension L of E, define $\mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}$ to be the limit-preserving stack over $\mathcal{O}/\varpi^a$ such that for each finite type $\mathcal{O}/\varpi^a$-algebra, $\mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}(A)$ classifies Breuil-Kisin-Fargues Gal_L-modules F^{inf} with A-coefficients and $\widehat{G}$-structure which admits all descents over L such that $F^{\mathrm{inf}} \times^{\widehat{G}} \mathrm{GL}_d$ is of height h and is equipped with canonical Gal_L-action in the sense of Proposition 2.3.3.

2.2.8. Lemma Let $\pi_L^\flat$ be a compatible system of p-power roots of a uniformizer π_L of L. The morphism

$$\mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h} \hookrightarrow \mathcal{C}^a_{L,\mathrm{GL}_d,\mathrm{ss},h} \underset{\mathcal{C}^a_{\pi_L^\flat,s,\mathrm{GL}_d,h}}{\times} \mathcal{C}^a_{\pi^\flat,s,\widehat{G},h} \tag{3}$$

is a closed immersion.

Proof. The morphism $\mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h} \to \mathcal{C}^a_{\pi^\flat,s,\widehat{G},h}$ is monomorphism by the definition of $\mathcal{C}^a_{\pi^\flat,s,\widehat{G},h}$. By [EG23, Proposition 4.5.17], (3) is a monomorphism. Let A be a finite type $\mathcal{O}/\varpi^a$-algebra. By Lemma 2.2.2, An object $F \in \mathcal{C}^a_{\pi^\flat,s,\widehat{G},h}(A)$ lies in the essential image of $\mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}(A)$ if and only if for any $x \in^f \mathrm{Rep}_{\widehat{G}}$,

$F \times^{\widehat{G}} x$ admits all descents over L when regarded as a Breuil-Kisin-Fargues Gal_L-module with A-coefficients. By the proof of [EG23, Proposition 4.5.17], the admitting all descents over L is a closed condition. $\square$

2.2.9. Lemma $\mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}$ is an algebraic stack of finite presentation over $\mathcal{O}/\varpi^a$, and have affine diagonals.

Proof. It follows from Lemma 2.1.8 and Lemma 2.2.8. $\square$

2.2.10. Lemma $\mathcal{C}_{L,\widehat{G},\mathrm{ss},h} := \varinjlim_a \mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}$ is a p-adic formal algebraic stack of finite presentation, and have affine diagonals. The morphism $\mathcal{C}_{L,\widehat{G},\mathrm{ss},h} \to \mathcal{X}_{L,\widehat{G}}$ is representable by algebraic spaces, proper and of finite presentation.

Proof. It is a generalization of [EG23, Theorem 4.5.20], and follows from Lemma 2.2.9 and [EG23, Proposition 4.5.17]. $\square$

2.3. Breuil-Kisin-Fargues lattices with ${}^LG$-structure

Fix a uniformizer $\pi = \pi_K$ of K.

2.3.1. Lemma If E is a tamely ramified Galois extension of K of ramification index e, then $\pi_E := \pi_K^{1/e}$ is a uniformizer of E.

Proof. We first show that $\pi_E \in E'$ for some unramified extension E' of E. Let $\Pi \in \mathcal{O}_E$ be an arbitrary uniformizer. We have $\Pi^e = c\pi$ for some $c \in \mathcal{O}_E^\times$. Write $\bar{c} \in \kappa_E$ for the image of c in the residue field κ_E and write $[\bar{c}]$ for the Teichmüller lift of $\bar{c}$. Since $[\bar{c}]$ admits an e-th root μ in an unramified extension of E, by replacing Π by $\Pi\mu$, we can assume $c = 1 + \pi x$ for some $x \in \mathcal{O}_E$. Thus $\frac{\Pi^e}{\pi_E^e} - 1 \in \Pi\mathcal{O}_E$. After multiplying Π by an e-th root of unity, we have $\frac{\Pi}{\pi_E} - 1 \in \Pi\mathcal{O}_E$, and thus $|\Pi - \pi_E| < |\Pi|$. By Krasner's lemma (see, for example, [L23B, Lemma 4.1.1]), we have $\pi_E \in E'[\Pi] = E'$.

Since $x^e = \pi$ is an Eisenstein polynomial, $K[\pi_E]$ is totally ramified over K. Therefore $\pi_E \in E$. $\square$

Fix a tame Galois extension E of K of ramification index e. By the lemma above, $\pi_E := \pi^{1/e}$ is a uniformizer of E. Let $\pi_E^\flat$ be a compatible system of p-power roots of π_E. Set $\pi^\flat := (\pi_E^\flat)^e$, which is a compatible system of p-power roots of π. Set $u := [\pi^\flat]$ and $u_E := [\pi_E^\flat]$. We have

$$\begin{aligned} \mathfrak{S}_K := \quad & \mathfrak{S}_{\pi^\flat} &= \kappa_K[[u]] \\ \mathfrak{S}_E := \quad & \mathfrak{S}_{\pi_E^\flat} &= \kappa_E[[u_E]]. \end{aligned}$$

It is clear that $\operatorname{Spec}\mathfrak{S}_E[1/u_E] \to \operatorname{Spec}\mathfrak{S}_K[1/u]$ is a Galois cover with Galois group canonically identified with $\mathrm{Gal}(E/K)$.

2.3.2. Semistable Galois representations and Breuil-Kisin-Fargues lattices

There exists an $\mathbb{A}_{\mathrm{inf}}$-linear isomorphism

$$\begin{aligned} A_{\mathrm{inf}} \underset{\mathfrak{S}_K}{\otimes} \mathfrak{S}_E \quad & \to \textstyle\prod_{\sigma \in \mathrm{Gal}(E/K)} A_{\mathrm{inf}} \\ a \otimes b \quad & \mapsto (a\sigma(b))_\sigma \end{aligned}$$

The diagonal map $\Delta : A_{\mathrm{inf}} \underset{\mathfrak{S}_K}{\otimes} \mathfrak{S}_E \to A_{\mathrm{inf}}$, $a \otimes b \mapsto ab$ is the unique Gal_E-equivariant A_{inf}-linear homomorphism such that the composition

$$A_{\mathrm{inf}} \xrightarrow{a \mapsto a\otimes 1} A_{\mathrm{inf}} \underset{\mathfrak{S}_K}{\otimes} \mathfrak{S}_E \xrightarrow{\Delta} A_{\mathrm{inf}}$$

is the identity map. Recall that $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}^\flat})$; we also need to consider the base-changed version

$$W(\mathbb{C}^\flat) \xrightarrow{a \mapsto a\otimes 1} W(\mathbb{C}^\flat) \underset{\mathfrak{S}_K}{\otimes} \mathfrak{S}_E = W(\mathbb{C}^\flat) \underset{\mathbb{A}_K}{\otimes} \mathbb{A}_E \xrightarrow{\Delta} W(\mathbb{C}^\flat).$$

Let (F, ϕ_F, ρ_F) be an étale $(\varphi, \mathrm{Gal}_K)$-module with ${}^L G$-structure over $W(\mathbb{C}^\flat)$ that corresponds to an L-parameter. By [L23B, Subsection 1.2], we have a $\mathrm{Gal}(E/K)$-equivariant isomorphism

$$\bar{F} := F \times^{{}^L G} \mathrm{Gal}(E/K) \cong \mathrm{Spec}\, W(\mathbb{C}^\flat) \otimes_{\mathbb{A}_K} \mathbb{A}_E \tag{4}$$

which we fix once for all. Moreover $F \to \bar{F}$ defines a $\widehat{G}$-torsor over $\bar{F}$. Consider the pullback diagram

$$\begin{tikzcd} F_\Delta \arrow[r, hook] \arrow[d] & F \arrow[d] \\ \mathrm{Spec}\, W(\mathbb{C}^\flat) \arrow[r, hook, "\Delta"] & \bar{F} \end{tikzcd}$$

Since Δ is an Gal_E-equivariant embedding, $F_\Delta \hookrightarrow F$ is both φ-equivariant and Gal_E-equivariant; therefore F_Δ inherits a structure of étale $(\varphi, \mathrm{Gal}_E)$-module with $\widehat{G}$-structure, which we write as $(F_\Delta, \phi_\Delta, \rho_\Delta)$ for simplicity.

Now assume the étale $(\varphi, \mathrm{Gal}_K)$-module (F, ϕ_F, ρ_F) with ${}^L G$-structure is potentially semistable; suppose it becomes semistable after restricting to Gal_L for some finite Galois extension L of E. By Proposition 2.3.3, the semistable étale $(\varphi, \mathrm{Gal}_L)$-module $(F_\Delta, \phi_\Delta, \rho_\Delta|_{\mathrm{Gal}_L})$ admits a unique Breuil-Kisin-Fargues Gal_L-lattice F_Δ^{inf} with $\widehat{G}$-structure. Note that F_Δ^{inf} is Gal_E-invariant (c.f. [EG23, Corollary F.23]), and F_Δ^{inf} is indeed a Breuil-Kisin-Fargues Gal_E-module with $\widehat{G}$-structure. Consider the following pushout diagram

$$\begin{tikzcd} F_\Delta \arrow[r, hook] \arrow[d] & F \arrow[r, equal] & \coprod_{\sigma \in \mathrm{Gal}(E/K)} \sigma(F_\Delta) \arrow[d] \\ F_\Delta^{\mathrm{inf}} \arrow[rr, hook] & & \coprod_{\sigma \in \mathrm{Gal}(E/K)} \sigma(F_\Delta^{\mathrm{inf}}) \end{tikzcd}$$

Set $F^{\mathrm{inf}} := \coprod_{\sigma \in \mathrm{Gal}(E/K)} \sigma(F_\Delta^{\mathrm{inf}})$. Concretely, $\mathcal{O}_{F_\Delta^{\mathrm{inf}}} \subset \mathcal{O}_{F_\Delta}$ (the structure sheaf) is an $\mathbb{A}_{\mathrm{inf}}$-submodule, and $\mathcal{O}_{F^{\mathrm{inf}}}$ is by definition the sum of the $\mathrm{Gal}(E/K)$-translations of $\mathcal{O}_{F_\Delta^{\mathrm{inf}}}$ in $\mathcal{O}_F$ (note that F_Δ is a connected component of F and $\mathcal{O}_{F_\Delta}$ is a direct summand of $\mathcal{O}_F$).

2.3.3. Proposition Let F be an étale $(\varphi, \mathrm{Gal}_K)$-module with ${}^L G$-structure. Assume $V(F)$ is a potentially semistable ${}^L G$-valued L-parameter. Then there exists a unique Breuil-Kisin-Fargues Gal_K-module F^{inf} with ${}^L G$-structure which admits all descents over L and which satisfies $F = F^{\mathrm{inf}} \otimes_{\mathbb{A}_{\mathrm{inf}}} W(\mathbb{C}^\flat)$.

Proof. By the construction above, $\mathcal{O}_{F^{\mathrm{inf}}}$ is stable under the $\mathcal{O}_{{}^LG}$-coaction on $\mathcal{O}_F$. In particular, F^{inf} is an ${}^LG$-torsor over $\operatorname{Spec}\mathbb{A}_{\mathrm{inf}}$.

It remains to show F^{inf} is Gal_K-stable. It follows from Tannakian formalism and the proof of [EG23, Corollary F.23]. □

2.3.4. Remark Given the Breuil-Kisin-Fargues Gal_K-lattice F^{inf}, we can recover the Breuil-Kisin-Fargues Gal_E-lattice F^{inf}_Δ. Consider

$$\bar{F}^{\mathrm{inf}} := F^{\mathrm{inf}} \times^{{}^LG} \mathrm{Gal}(E/K).$$

We have $\bar{F}^{\mathrm{inf}} \cong \operatorname{Spec}\mathbb{A}\otimes_{\mathfrak{S}_K}\mathfrak{S}_E$, and such an identification is uniquely determined by Equation (4). The following Cartesian diagram

$$\begin{tikzcd} F^{\mathrm{inf}}_\Delta \arrow[r, hook] \arrow[d] & F^{\mathrm{inf}} \arrow[d] \\ \operatorname{Spec}\mathbb{A}_{\mathrm{inf}} \arrow[r, hook, "\Delta"] & \bar{F}^{\mathrm{inf}} \end{tikzcd}$$

recovers F^{inf}_Δ from F^{inf}.

2.3.5. Definition For each Galois extension L/F, define $\mathcal{C}^a_{\mathrm{BKF}-L,{}^LG}$ to be the limit-preserving stack over $\mathcal{O}/\varpi^a$ such that $\mathcal{C}^a_{\mathrm{BKF}-L,{}^LG}(A)$ is the groupoid of pairs (F^{inf},c) where F^{inf} is a Breuil-Kisin-Fargues Gal_L-module with ${}^LG$-structure with A-coefficients and c is an Gal_L-equivariant morphism $F^{\mathrm{inf}}\times^{{}^LG}\mathrm{Gal}(E/K)\cong \operatorname{Spec}\mathbb{A}_{\mathrm{inf},A}\otimes_{\mathfrak{S}_K}\mathfrak{S}_E$, for all finite type $\mathcal{O}/\varpi^a$-algebras A.

If $L\supset E$, there exists a canonical morphism

$$\mathcal{C}^a_{\mathrm{BKF}-K,{}^LG} \to \mathcal{C}^a_{\mathrm{BKF}-L,\widehat{G}}$$

defined by sending (F^{inf},c) to the pullback of F^{inf} along $c^{-1}\circ\Delta : \operatorname{Spec}\mathbb{A}_{\mathrm{inf},A}\to F\times^{{}^LG}\mathrm{Gal}(E/K)$.

Set

$$\mathcal{C}^{L/K,a}_{{}^LG,\mathrm{ss},h} := \mathcal{C}^a_{\mathrm{BKF}-K,{}^LG} \underset{\mathcal{C}^a_{\mathrm{BKF}-L,\widehat{G}}}{\times} \mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}$$

and

$$\mathcal{C}^{L/K}_{{}^LG,\mathrm{ss},h} := \varinjlim_a \mathcal{C}^{L/K,a}_{{}^LG,\mathrm{ss},h}$$

for all Galois extensions L/E.

2.3.6. Lemma The morphism

$$\mathcal{C}^{L/K,a}_{{}^LG,\mathrm{ss},h} \hookrightarrow \mathcal{X}_{K,{}^LG} \underset{\mathcal{X}_{L,\widehat{G}}}{\times} \mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}$$

is a closed immersion.

Proof. By Remark 2.3.4 and the construction before Proposition 2.3.3, we see the morphism is a monomorphism. $\mathcal{C}^{L/K,a}_{{}^LG,\mathrm{ss},h}$ is the Gal_K-stable locus of $\mathcal{X}_{K,{}^LG} \underset{\mathcal{X}_{L,\widehat{G}}}{\times} \mathcal{C}^a_{L,\widehat{G},\mathrm{ss},h}$, which is a closed condition. □

2.3.7. Lemma For any Galois extension L/E, $\mathcal{C}^{L/K}_{{}^LG,\mathrm{ss},h}$ is a p-adic formal algebraic stack of finite presentation with affine diagonal. The forgetful morphism $\mathcal{C}^{L/K}_{{}^LG,\mathrm{ss},h} \to \mathcal{X}_{K,{}^LG}$ is representable by algebraic spaces, proper and of finite presentation.

Proof. Combine Lemma 2.2.10 and Lemma 2.3.6. □

2.4. Inertial types Let A° be a p-adically complete flat $\mathcal{O}$-algebra which is topologically of finite type over $\mathcal{O}$, and write $A := A^\circ[1/p]$.

Let L/K be a finite Galois extension containing E with inertia group $I_{L/K}$ and suppose $\mathcal{O}[1/p]$ contains the image of all embeddings $L \hookrightarrow \overline{\mathbb{Q}}_p$ and that all irreducible $\mathcal{O}[1/p]$-representations of $I_{L/K}$ are absolutely irreducible. Write l for the residue field of L and write $L_0 = W(l)[1/p]$.

In [EG23, Section 4.6], Weil-Deligne representations $\mathrm{WD}(\mathfrak{M}^{\mathrm{inf}})$ are attached to Breuil-Kisin-Fargues Gal_K modules $\mathfrak{M}^{\mathrm{inf}}$ with A°-coefficients that admits all descents over L. The underlying A-module of $\mathrm{WD}(\mathfrak{M}^{\mathrm{inf}})$ is $e_\sigma(\overline{\mathfrak{M}}_{A^\circ} \otimes_{A^\circ} A)$ and it is equipped with an A-linear action of $I_{L/K}$ (see loc. cit. for unfamiliar notations). Here $e_\sigma \in L_0 \otimes_{\mathbb{Q}_p} \mathcal{O}[1/p]$ is the idempotent corresponding to a fixed choice of embedding $\sigma : L_0 \hookrightarrow \mathcal{O}[1/p]$.

2.4.1. Lemma and Definition Let A° be a p-adically complete flat $\mathcal{O}$-algebra which is topologically of finite type over $\mathcal{O}$, and write $A := A^\circ[1/p]$.

Let F^{inf} be a Breuil-Kisin-Fargues Gal_K modules $\mathfrak{M}^{\mathrm{inf}}$ with A°-coefficients and ${}^LG$-structure that admits all descents over L. The functor $\mathrm{WD}(F^{\mathrm{inf}})$

$$
\begin{aligned}
{}^f\,\mathrm{Rep}_{{}^LG} &\to \mathrm{Vect}_A \\
x &\mapsto \mathrm{WD}(F^{\mathrm{inf}} \times^{{}^LG} x)
\end{aligned}
$$

is a faithful, exact, symmetric monoidal functor.

Proof. The functor $\mathrm{WD}(F^{\mathrm{inf}})$ is clearly lax monoidal. Strict monoidality, faithfulness and exactness are local properties. Since A° is $\mathcal{O}$-flat and topologically of finite type over $\mathcal{O}$, it suffices to check the exactness (and monoidality and faithfulness) of $\mathrm{WD}(F^{\mathrm{inf}})$ at Λ-points for finite flat $\mathcal{O}$-algebras Λ (because the union of images of Λ-points of $\mathrm{Spec}\, A^\circ$ covers all closed points of $\mathrm{Spec}\, A^\circ$). As is observed in [EG23, Remark 4.6.2], in the finite $\mathcal{O}$-flat coefficients situation, $\mathrm{WD}(-)$ can be identified with Fontaine's D_{pst} functor, which is well-known to be exact (see also [EG23, Section F.24]); faithfulness and monoidality are clear. □

2.4.2. Lemma Let A° be a p-adically complete flat $\mathcal{O}$-algebra which is topologically of finite type over $\mathcal{O}$, and write $A := A^\circ[1/p]$.

Let F^{inf} be a Breuil-Kisin-Fargues Gal_K modules $\mathfrak{M}^{\mathrm{inf}}$ with A°-coefficients and ${}^LG$-structure that admits all descents over L. Then $\mathrm{WD}(F^{\mathrm{inf}})$ is a ${}^LG$-torsor over $\mathrm{Spec}\, A$ equipped with an action of I_F, whose formation is compatible with base change $A^\circ \to B^\circ$ of p-adically complete $\mathcal{O}$-flat algebras which are topologically of finite type over $\mathcal{O}$.

Proof. Combine [EG23, Proposition 4.6.3] and Lemma 2.4.1. □

2.4.3. Definition Let τ be an ${}^LG(\mathcal{O}[1/p])$-valued representation of $I_{L/K}$. In the setting of Lemma 2.4.2, we say F^{inf} has inertial type τ if étale locally on $\mathrm{Spec}\, A$, $\mathrm{WD}(F^{\mathrm{inf}})$ is isomorphic to the base change to A of τ.

2.4.4. Lemma Fix a finite Galois extension L/E. There are finitely many $\widehat{G}$-conjugacy classes of inertial types $I_{L/K} \to {}^L G(\bar{\mathbb{Q}}_p)$.

Proof. Since $I_{L/K}$ is a finite group, and $\bar{\mathbb{Q}}_p$ is a characteristic 0 field, all group homomorphisms $I_{L/K} \to {}^L G(\bar{\mathbb{Q}}_p[\varepsilon]/\varepsilon^2)$ factor through ${}^L G(\bar{\mathbb{Q}}_p)$: embed ${}^L G$ in GL_N for some N, if $g \in I_{L/K}$ is sent to $x + \varepsilon y$, then $(x + \varepsilon y)^n = x^n = 1$ for some positive integer n, and thus $y = 0$. Therefore, deformation theory is trivial in our context.

By the proof of Lemma 2.4.5 below (which makes use of only affine GIT theory and there is no circular reasoning), the coarse moduli space of all $\widehat{G}$-conjugacy classes of inertial types $I_{L/K} \to {}^L G(\bar{\mathbb{Q}}_p)$ is a finite type scheme over $\bar{\mathbb{Q}}_p$. So we are done. □

2.4.5. Lemma In the setting of Lemma 2.4.2, we can decompose $\operatorname{Spec} A$ as the disjoint union of open and closed subschemes $\operatorname{Spec} A^\tau$, where $\operatorname{Spec} A^\tau$ is the locus over which F^{inf} has inertial type τ. Furthermore, the formation of this decomposition is compatible with base change $A^\circ \to B^\circ$ of p-adically complete $\mathcal{O}$-flat algebras which are topologically of finite type over $\mathcal{O}$.

Proof. We start with analyzing the (coarse) moduli of inertial types. Say $I_{L/K} = \{x_1, \dots, x_N\}$ consists of N elements. Consider the conjugation action of $\widehat{G}_{\bar{\mathbb{Q}}_p}$ on the N-tuple

$$
{}^L G^{(N)}_{\bar{\mathbb{Q}}_p} := {}^L G_{\bar{\mathbb{Q}}_p} \times \cdots \times {}^L G_{\bar{\mathbb{Q}}_p}
$$

Write ${}^L G^{(N)}_{\bar{\mathbb{Q}}_p} /\!\!/ \widehat{G}_{\bar{\mathbb{Q}}_p}$ for the GIT quotient. By [BMRT11, Theorem 1.1] (which allows disconnected groups), if $x_1, \dots, x_N$ generate a finite subgroup of ${}^L G(\bar{\mathbb{Q}}_p)$, then the orbit $\widehat{G}_{\bar{\mathbb{Q}}_p} \cdot (x_1, \dots, x_N)$ is closed in ${}^L G^{(N)}_{\bar{\mathbb{Q}}_p}$. Since the affine GIT quotient is a *good quotient*, the image of $\widehat{G}_{\bar{\mathbb{Q}}_p} \cdot (x_1, \dots, x_N)$ in the GIT quotient is a closed point. Inertial types $I_{L/K} \to {}^L G(\bar{\mathbb{Q}}_p)$ corresponds to tuples $(x_1, \dots, x_N)$ that satisfy a finite number of equations imposed by the group laws; write $X \subset {}^L G^{(N)}_{\bar{\mathbb{Q}}_p}$ for the closed affine subscheme corresponding to inertial types. For each $(x_1, \dots, x_N) \in X$, $\{x_1, \dots, x_n\}$ generate a finite subgroup, and thus $X /\!\!/ \widehat{G}_{\bar{\mathbb{Q}}_p}$ is an orbit space. Since there are only finitely many conjugacy classes of inertial types by Lemma 2.4.4, $X /\!\!/ \widehat{G}_{\bar{\mathbb{Q}}_p}$ is zero-dimensional and is thus a disjoint union of points. By descent, $X /\!\!/ \widehat{G}$ defined over $\mathcal{O}[1/p]$ is also a zero-dimensional affine variety. Since the formation of $WD(F^{\mathrm{inf}})$ is compatible with base change, the functor of points interpretation yields a canonical morphism $\operatorname{Spec} A \to [X/\widehat{G}] \to X /\!\!/ \widehat{G}$. The decomposition $\operatorname{Spec} A = \amalg \operatorname{Spec} A^\tau$ is the base change of the corresponding decomposition on $X /\!\!/ \widehat{G}$. □

2.5. p-adic Hodge types Next we analyze p-adic Hodge types. By the geometric Shapiro's lemma [L23B, Proposition 7.2.4], there is no difference in working with G or $\mathrm{Res}_{K/\mathbb{Q}_p} G$. For ease of notation, we will often replace G by $\mathrm{Res}_{K/\mathbb{Q}_p} G$ and insist $K = \mathbb{Q}_p$.

Recall the following characterization of cocharacters.

2.5.1. Lemma ([Ba12, Lemma 3.0.10]) Let H be a split connected reductive group over E and let $\mu,\ \mu'$ be two cocharacters of $H_{\bar{\mathbb{Q}}_p}$ defined over E. If for any algebraic representation $x : H \to \mathrm{GL}(V)$, cocharatcers $x \circ \mu$ and $x \circ \mu'$ are conjugate by an element of $\mathrm{GL}(V)(\bar{\mathbb{Q}}_p)$, then $\mu,\ \mu'$ are conjugate by an element of $H(\bar{\mathbb{Q}}_p)$.

2.5.2. Corollary Let H be a split connected reductive group over E. Let F be a trivial H-torsor over E. Let $\eta,\ \eta'$ be two exact $\otimes$-filtrations on F. If for any algebraic representation $x : H \to \mathrm{GL}(V)$, cocharatcers $x \circ \eta$ and $x \circ \eta'$ are conjugate by an element of $\mathrm{GL}(V)(\bar{\mathbb{Q}}_p)$, then $\eta,\ \eta'$ are conjugate by an element of $H(\bar{\mathbb{Q}}_p)$.

Proof. By [SN72, IV.2.4], both $\eta,\ \eta'$ are splittable exact $\otimes$-filtrations, that is, they both are the canonical filtrations attached to exact $\otimes$-gradings $\widetilde{\eta}, \widetilde{\eta}'$ on F. An exact $\otimes$-grading $\widetilde{\eta}$ is equivalent to a cocharacter $\mu : \mathrm{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(F)$. Since F is a trivial torsor, a choice of trivialization induces a group scheme isomorphism $\underline{\mathrm{Aut}}^{\otimes}(F) \cong H$ and the choice of trivialization does not affect the conjugacy class of the composition $\mathrm{G}_m \xrightarrow{\widetilde{\eta}} \underline{\mathrm{Aut}}^{\otimes}(F) \cong H$, which we also denote by $\widetilde{\eta}$ by abuse of notation.

We remark that two filtrations $\eta,\ \eta'$ are H-conjugate if and only if two corresponding cocharacters $\mu,\ \mu'$ are H-conjugate. Indeed, two cocharacters $\mu_1,\ \mu_2$ induce the same filtration if and only if they are conjugate by an element of $\underline{\mathrm{Aut}}^{\otimes !}_{\mu}(F)$ which is a closed subgroup scheme of $\underline{\mathrm{Aut}}^{\otimes}(F)$. See [BG19, Section 2.7] for unfamiliar notations.

The corollary now follows from Lemma 2.5.1. $\square$

2.5.3. Conjugacy classes of filtrations and Hodge types Recall that in [EG23, Definition 4.7.7], A *Hodge type of rank d* $\underline{\lambda}$ is defined to be a set of tuples of integers $\{\lambda_{\sigma,j}\}_{\sigma:K\hookrightarrow\bar{\mathbb{Q}}_p, 1\le j\le d}$ with $\lambda_{\sigma,j} \geq \lambda_{\sigma,j+1}$. It is clear that rank-$d$ Hodge types $\underline{\lambda}$ are in one-to-one correspondence with $\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}}\mathrm{GL}_d$-conjugacy classes of cocharacters of $\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}}\mathrm{GL}_d$.

So Hodge types for G should be defined as conjugacy classes of cocharacters of $\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}} G$. As is explained at the beginning of this subsection, it is harmless to replace G by $\mathrm{Res}_{K/\mathbb{Q}_p} G$, so Hodge types become conjugacy classes of cocharacters of $\widehat{\mathrm{Res}_{K/\mathbb{Q}_p}} G$.

2.5.4. Definition Let A° be a p-adically complete flat $\mathcal{O}$-algebra which is topologically of finite type over $\mathcal{O}$, and let F^{inf} be a Breuil-Kisin-Fargues Gal_E-module with $\widehat{G}$-structure of height at most h, which admits all descents over some finite extension L of E.

Let $\underline{\lambda} : \mathrm{G}_m \to \widehat{G}_{\bar{\mathbb{Q}}_p}$ of a cocharacter of $\widehat{G}_{\bar{\mathbb{Q}}_p}$. We say F^{inf} has *Hodge type* $\underline{\lambda}$ if for all algebraic representations $f : \widehat{G} \to \mathrm{GL}(V)$, $F^{\mathrm{inf}} \times^{\widehat{G}} V$ has Hodge type $\underline{\lambda} \circ f$ in the sense of [EG23, Corollary 4.7.8].

2.5.5. Lemma Fix a number $C > 0$. Fix an embedding $i : \widehat{G} \hookrightarrow \mathrm{GL}_N$. There exists only finitely many conjugacy classes of cocharacters $\underline{\lambda}$ of $\widehat{G}_{\bar{\mathbb{Q}}_p}$ such that $\underline{\lambda} \circ i$ correspond to a tuple of integers $\{\lambda_{i,1}, \dots, \lambda_{i,N}\}$ such that each $\lambda_{i,\bullet}$ has absolute value bounded by C.

Proof. Clear. $\square$

2.5.6. Lemma In the context of Definition 2.5.4, we can write $\mathrm{Spec}\, A$ as a disjoint union of open and closed substacks $\mathrm{Spec}\, A^{\underline{\lambda}}$ over which F^{inf} is of Hodge type $\underline{\lambda}$. Moreover, this decomposition is compatible with base change $A^\circ \to B^\circ$ of p-adically complete flat $\mathcal{O}$-algebras which are topologically of finite type over $\mathcal{O}$.

Proof. By Corollary 2.5.2, for any two non-equivalent Hodge types $\underline{\lambda}$ and $\underline{\lambda}'$, there exists an algebraic representation f such that $\underline{\lambda} \circ f$ and $\underline{\lambda} \circ f$ are not conjugate.

Since $\operatorname{Spec} A$ is topologically of finite type, there are only finitely many Hodge types occuring on $\operatorname{Spec} A$ (by Lemma 2.5.5, the claim reduces to the GL_N-case, which is well-known). In particular, we can choose finitely many different algebraic representations $f_1, \ldots, f_m$ that distinguish all Hodge types occuring on $\operatorname{Spec} A$, in the sense that for any two non-equivalent Hodge types $\underline{\lambda}$ and $\underline{\lambda}'$, there exists an algebraic representation f_i $(1 \le i \le m)$ such that $\underline{\lambda} \circ f_i$ and $\underline{\lambda} \circ f_i$ are not conjugate.

The lemma has thus been reduced to the GL_n-case, which is dealt with in [EG23, Corollary 4.7.8], since $\operatorname{Spec} A^{\underline{\lambda}}$ is the intersection of $\operatorname{Spec} A^{\underline{\lambda}\circ f_i}$, $1 \le i \le m$. □

2.5.7. Corollary In the context of Definition 2.5.4, we can write

$$\operatorname{Spec} A = \coprod_{\underline{\lambda},\tau} \operatorname{Spec} A^{\underline{\lambda},\tau}$$

where $\operatorname{Spec} A^{\underline{\lambda},\tau}$ is the locus over which F^{inf} is of Hodge type $\underline{\lambda}$ and inertial type τ. Moreover, this decomposition is compatible with base change $A^\circ \to B^\circ$ of p-adically complete flat $\mathcal{O}$-algebras which are topologically of finite type over $\mathcal{O}$.

Proof. Combine Lemma 2.5.6 and Lemma 2.4.5. □

2.5.8. Remark (1) By Lemma 2.5.1, there is an intrinsic characterization of Hodge types for Breuil-Kisin-Fargues modules with $\widehat{G}$-structure. The Hodge filtration on the de Rham periods ([EG23, Definition 4.7.6]) is exact and $\otimes$-compatible, and thus defines an exact $\otimes$-filtration μ on $D_{\mathrm{dR}}(F^{\mathrm{inf}})$. The clopen $\operatorname{Spec} A^{\underline{\lambda}}$ in Lemma 2.5.6 is precisely the locus of $\operatorname{Spec} A$ where μ is conjugate to $\underline{\lambda}$ at all closed points.

(2) The formation of Hodge types is not sensitive to restriction of fields. So we say a Breuil-Kisin-Fargues Gal_K module with ${}^LG$-structure has Hodge type $\underline{\lambda}$ if when restricted to Gal_E, it is a Breuil-Kisin-Fargues Gal_E module with $\widehat{G}$-structure and Hodge type $\underline{\lambda}$.

2.5.9. Definition Write $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h}$ for the maximal $\mathcal{O}$-flat substack of $\mathcal{C}^{L/K}_{{}^LG,\mathrm{ss},h}$ (see [EG23, Appendix A] for the existence). We record the following proposition.

2.5.10. Proposition Let L/E be a finite Galois extension. Then $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h}$ is the scheme-theoretical union of closed substacks $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h,\underline{\lambda},\tau}$, which is uniquely characterized by the following property: if A° is a finite flat $\mathcal{O}$-algebra, then an A°-point of $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h}$ factors through $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h,\underline{\lambda},\tau}$ if and only if the corresponding Breuil-Kisin-Fargues F^{inf} has Hodge type $\underline{\lambda}$ and inertial type τ.

Moreover, $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h,\underline{\lambda},\tau}$ is a p-adic formal algebraic stack of finite presentation which is flat over $\operatorname{Spf} \mathcal{O}$ and whose diagonal is affine. The natural morphism $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h,\underline{\lambda},\tau} \to \mathcal{X}_{K,{}^LG}$ is representable by algebraic spaces, proper, and of finite presentation.

Proof. The proof is formally identical to that of [EG23, Proposition 4.8.2]. □

2.5.11. Definition Let τ be an inertial type and let $\underline{\lambda}$ be a Hodge type. Let h be a sufficiently large integer such that $\underline{\lambda} \circ i$ is bounded by h (where i is the fixed embedding $\widehat{G} \to \mathrm{GL}_d$), and let L/E be a Galois extension such that τ is trivial when restricted to I_L. Define $\mathcal{X}^{\mathrm{ss},\lambda,\tau}_{K,{}^LG}$ to be the scheme-theoretic image of $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^LG,\mathrm{ss},h,\underline{\lambda},\tau}$ in $\mathcal{X}_{K,{}^LG}$.

2.6. Potentially semistable deformation rings Recall the following theorem about Galois deformation rings.

2.6.1. Theorem ([BG19, Theorem A], [Ba12, Theorem 3.0.12]) Let τ be an inertial type, and let $\underline{\lambda}$ be a Hodge type. Fix a mod p L-parameter $\bar{\rho} : \mathrm{Gal}_K \to {}^L G(\mathbb{F})$. The framed potentially semistable deformation ring $R_{\bar{\rho}}^{\square,\tau,\underline{\lambda}}$ is equidimensional of dimension

$$1 + \dim_{\mathbb{Q}_p} \widehat{G} + \dim_{\mathbb{Q}_p} \widehat{G}/P_{\underline{\lambda}}$$

where $P_{\underline{\lambda}} := \underline{\mathrm{Aut}}^{\otimes}(\underline{\lambda})$ is the parabolic subgroup of $\widehat{G}$ which stabilizes the exact $\otimes$-filtration associated to v. (See [BG19, Section 2.7] for unfamiliar notations.)

Denote by $R_{\bar{\rho}}^{\square}$ the universal deformation ring. The potentially semistable deformation ring $R_{\bar{\rho}}^{\square,\tau,\underline{\lambda}}$ is the unique $\mathcal{O}$-flat quotient such that for any finite local $\mathcal{O}[1/p]$-algebra B, any $\mathcal{O}[1/p]$-homomorphism $\zeta : R_{\bar{\rho}}^{\square} \to B$ factors through $R_{\bar{\rho}}^{\square,\tau,\underline{\lambda}}$ if and only if ζ corresponds to an L-parameter $\rho_\zeta : \mathrm{Gal}_K \to {}^L G(B)$ that is potentially semi-stable of inertial type τ and Hodge type $\underline{\lambda}$.

Proof. The first paragraph is [BG19, Theorem A]; the second paragraph is [Ba12, Theorem 3.0.12]. We remark that in [BG19], any non-split group G is allowed. In [Ba12], only split groups G are considered. However, the same argument works through as long as inertial types can be constructed, which we have done in Subsection 2.4. We also remark that [BG19] directly uses results of [Ba12] even though the setting of [BG19] is more general. $\square$

2.6.2. Lemma Assume $E/\mathbb{Q}_p$ is tame. In the setting of Theorem 2.6.1, the morphism $\mathrm{Spf}\, R_{\bar{\rho}}^{\square,\tau,\underline{\lambda}} \to \mathcal{X}_{K,{}^L G}$ factors through a versal morphism

$$\mathrm{Spf}\, R_{\bar{\rho}}^{\square,\tau,\underline{\lambda}} \to \mathcal{X}_{K,{}^L G}^{\mathrm{ss},\tau,\underline{\lambda}}.$$

Proof. The proof is formally identical to that of [EG23, Proposition 4.8.10]. The inputs are

- Algebraicity of $\mathcal{X}_{K,{}^K G}$ (the main theorem of [L23B]);
- Description of finite $\mathcal{O}$-flat points of $\mathcal{C}^{L/K,\mathrm{fl}}_{{}^L G,\mathrm{ss},h,\underline{\lambda},\tau}$ (Proposition 2.5.10);
- Description of finite $\mathcal{O}$-flat points of $\mathrm{Spf}\, R_{\bar{\rho}}^{\square,\tau,\underline{\lambda}}$ (Theorem 2.6.1); and
- Existence of Breuil-Kisin-Fargues lattices.

The last bullet point in the GL_d-case is [EG23, Theorem 4.7.13], which holds for general (split) groups G, as is observed in the thesis of B. Levin (see the proof of [Lev13, Proposition 5.4.2]). The non-split group case follows from the split group case (see the proof of Proposition 2.3.3). $\square$

2.6.3. Theorem Assume $E/\mathbb{Q}_p$ is tame. Let τ be an inertial type and let $\underline{\lambda}$ be a Hodge type. Then $\mathcal{X}_{K,{}^L G}^{\mathrm{ss},\tau,\underline{\lambda}}$ is a p-adic formal algebraic stack which is of finite type and flat over $\mathrm{Spf}\,\mathcal{O}$. It is uniquely determined as the $\mathcal{O}$-flat closed substack of $\mathcal{X}_{K,{}^L G}$ by the following property: if A° is a finite $\mathcal{O}$-flat algebra, then $\mathcal{X}_{K,{}^L G}^{\mathrm{ss},\tau,\underline{\lambda}}(A^\circ)$ is the subgroupoid consisting of L-parameters which become potentially semistable of Hodge type $\underline{\lambda}$ and inertia type τ after inverting p.

The mod p fiber

$$\mathcal{X}_{K,{}^L G}^{\mathrm{ss},\tau,\underline{\lambda}} \underset{\mathrm{Spf}\,\mathcal{O}}{\times} \mathrm{Spec}\,\mathbb{F}$$

is equidimensional of dimension $\dim_{\mathbb{Q}_p} \widehat{G}/P_{\underline{\lambda}}$.

Proof. The proof is formally identical to that of [EG23, Theorem 4.8.12]. The first claim follows from Proposition 2.5.10, [L23B, Theorem 2] and [L23B, Proposition A.21]. The description of finite $\mathcal{O}$-flat points $\mathcal{X}^{\mathrm{ss},\tau,\underline{\lambda}}_{K,{}^LG}(A^\circ)$ follows from Lemma 2.6.2. The uniqueness follows from [EG23, Proposition 4.8.6]. The dimension calculation follows from Theorem 2.6.1 and the versality of $\operatorname{Spf} R^{\square,\tau,\underline{\lambda}}_{\bar{\rho}}$ established in Lemma 2.6.2 (compare with [EG23, Theorem 4.8.14]). ☐

2.7. The semistable Shapiro's Lemma

2.7.1. Theorem Assume E is tame over $\mathbb{Q}_p$. Under the identification ([L23B, Proposition 7.2.4])

$$\mathrm{Sha} : \mathcal{X}_{K,{}^LG} \cong \mathcal{X}_{\mathbb{Q}_p,{}^L\mathrm{Res}_{K/\mathbb{Q}_p} G},$$

we have

$$\mathcal{X}^{\mathrm{ss},\tau,\underline{\lambda}}_{K,{}^LG} = \mathcal{X}^{\mathrm{ss},\ \mathrm{Sha}(\tau),\ \mathrm{Sha}(\underline{\lambda})}_{\mathbb{Q}_p,{}^L\mathrm{Res}_{K/\mathbb{Q}_p} G}.$$

Proof. It follows from the uniqueness part of Theorem 2.6.3. ☐

2.8. Applications to tori

2.8.1. Lemma Let T be a torus over K. An L-parameter $\bar{\rho} : \mathrm{Gal}_K \to {}^LT(\bar{\mathbb{F}}_p)$ admits a lift $\bar{\rho} : \mathrm{Gal}_K \to {}^LT(W(\bar{\mathbb{F}}_p))$ which is potentially semistable of trivial Hodge type.

Proof. By the local Langlands correspondence for tori, there is a functorial bijection between L-parameters and continuous characters of $T(K)$. See [L23, Section 5.2]. A character $T(K) \to \bar{\mathbb{F}}_p^\times$ admits the Teichmüller lift $T(K) \to W(\bar{\mathbb{F}}_p)^\times$, which corresponds to a potentially semistable of trivial Hodge type. ☐

2.8.2. Corollary Let T be a torus over K which splits over a tame extension $E/\mathbb{Q}_p$. Write $\underline{0}$ for the trivial Hodge type. Then $\mathcal{X}_{{}^LT,\mathrm{red}}$ is the disjoint union of $\mathcal{X}^{\mathrm{ss},\tau,\underline{0}}_{{}^LT} \underset{\operatorname{Spf}\mathcal{O}}{\times} \operatorname{Spec}\mathbb{F}$ for various tame inertial types τ.

As a consequence, $\mathcal{X}_{{}^LT,\mathrm{red}}$ is equidimensional of dimension 0.

Proof. By Lemma 2.8.1, $\amalg_\tau \mathcal{X}^{\mathrm{ss},\tau,\underline{0}}_{{}^LT} \underset{\operatorname{Spf}\mathcal{O}}{\times} \operatorname{Spec}\mathbb{F} \to \mathcal{X}_{{}^LT,\mathrm{red}}$ is surjective. It remains to show the images of the various $\mathcal{X}^{\mathrm{ss},\tau,\underline{0}}_{{}^LT} \underset{\operatorname{Spf}\mathcal{O}}{\times} \operatorname{Spec}\mathbb{F}$ are disjoint. Since they are reduced algebraic stacks, it suffices to show their $\bar{\mathbb{F}}_p$-points are disjoint. Suppose $\bar{\rho}_1 \in \mathcal{X}^{\mathrm{ss},\tau_1,\underline{0}}_{{}^LT}(\bar{\mathbb{F}}_p)$ and $\bar{\rho}_2 \in \mathcal{X}^{\mathrm{ss},\tau_2,\underline{0}}_{{}^LT}(\bar{\mathbb{F}}_p)$. Write $[-]$ for the Teichmuller lift considered in the proof Lemma 2.8.1. We have $[\bar{\rho}_1] \in \mathcal{X}^{\mathrm{ss},\tau_1,\underline{0}}_{{}^LT}(W(\bar{\mathbb{F}}_p))$ and $[\bar{\rho}_2] \in \mathcal{X}^{\mathrm{ss},\tau_2,\underline{0}}_{{}^LT}(W(\bar{\mathbb{F}}_p))$. Since both $[\bar{\rho}_1]$ and $[\bar{\rho}_2]$ have finite image, we have $\mathrm{WD}([\bar{\rho}_1]) \cong \rho_1|_{W_K} \otimes_K \bar{\mathbb{Q}}_p$ and $\mathrm{WD}([\bar{\rho}_2]) \cong \rho_2|_{W_K} \otimes_K \bar{\mathbb{Q}}_p$. Thus $[\bar{\rho}_1|_{I_K}] = \tau_1$ and $[\bar{\rho}_2|_{I_K}] = \tau_2$, and $\bar{\rho}_1 = \bar{\rho}_2$ implies $\tau_1 = \tau_2$.

The second paragraph follows from Theorem 2.6.3. ☐

3. Remarks on disconnected reductive groups

In this section, we compile results on parabolic subgroups of disconnected groups for a lack of reference. We are specifically interested in groups that can be written as a semi-direct product of a connected reductive group and a finite group.

3.0.1. Lemma Let H be a connected reductive group over an algebraically closed field. The intersection of two Borels of H contains a maximal torus of H.

Proof. It follows immediately from the Bruhat decomposition. Fix a Borel pair (B,T) of H. Since all Borels are conjugate, any Borel B' can be written as gBg^{-1}, $g \in H$. Write $g = b_1 w b_2$ where $b_1, b_2 \in B$ and $w \in N_H(T)$. We have $B \cap B' = b_1(wBw^{-1} \cap B)b_1^{-1}$. It is clear that $b_1 T b_1^{-1} \subset B \cap B'$. □

3.0.2. Corollary Let H be a connected reductive group over $\bar{\mathbb{F}}_p$. Let Γ be a finite group of prime-to-p order acting on H.

Then two Γ-stable Borels of H have a common Γ-stable maximal torus.

Proof. Let B and B' be Γ-stable Borels. By Lemma 3.0.1, there exists a maximal torus $T \subset B \cap B'$.

Pick $\gamma \in \Gamma$. We have $\gamma(T) = uTu^{-1}$ for some $u \in U$ (the unipotent radical of B) since all maximal tori of B are conjugate to each other. Write s for the order of γ. We have $\gamma^s(T) = u^s T u^{-s} = T$. Thus $u^s \in N_H(T) \cap B = T$. On the other hand, the unipotent radical U is a (union of finite) p-group(s). Thus $u^{p^n} = 1$ for some positive integer n. By Bézout's Lemma, $1 = as + bp^n$ for $a, b \in \mathbb{Z}$. We have $u = (u^s)^a \in T$, and thus $\gamma(T) = T$. □

3.0.3. Lemma Let H be a connected reductive group over an algebraically closed field. Let $M \subset H$ be a maximal proper Levi subgroup. Then there are exactly two parabolics P, Q of H containing M, and $P \cap Q = M$.

Proof. Let $T \subset M$ be a maximal torus and let $\{\alpha_1, \dots, \alpha_r\}$ be a base of the roots $R(H,T)$. Write $M = Z_H(\lambda)$ for some cocharacter $\lambda : \mathrm{G}_m \to T$. Since M is a maximal proper Levi, $\langle \lambda, \alpha_i \rangle = 0$ for all but one α_i. Let's say $\langle \lambda, \alpha_1 \rangle \neq 0$. A parabolic P of H contains M if and only of P is of the form $P_H(\mu)$ for some cocharacter $\mu : \mathrm{G}_m \to T$ such that $\langle \mu, \alpha_i \rangle = 0$ for all $i > 1$ and $\langle \mu, \alpha_1 \rangle \neq 0$.

After possibly replacing μ by μ^{-1}, we assume $n_\mu := \langle \mu, \alpha_1 \rangle$ and $n_\lambda := \langle \lambda, \alpha_1 \rangle$ both have positive signs. Since $n_\mu \lambda$ and $n_\lambda \mu$ differ by a central cocharacter, we have $P_H(\lambda) = P_H(n_\mu \lambda) = P_H(n_\lambda \mu) = P_H(\mu)$.

Thus $P_H(\lambda)$ and $P_H(\lambda^{-1})$ are the only two parabolics containing M. □

The lemma above has the following extension.

3.0.4. Lemma Let H be a connected reductive group over an algebraically closed field, equipped with a pinning $(B, T, \{X_\alpha\})$. Let Γ be a finite group acting the pinned group $(H, B, T, \{X_\alpha\})$.

Let $M \subset H$ be a maximal proper Γ-stable Levi subgroup. Then there are exactly two parabolics P, Q of H containing M, and $P \cap Q = M$.

Proof. While maximal proper Levi's of H corresponds to elements of the base $\Delta(B,T)$, maximal proper Γ-stable Levi's of H corresponds to Γ-orbits of the base $\Delta(B,T)$. The rest of the proof is the same as that of Lemma 3.0.3. The two parabolics corresponds to the sign of the cocharacter $\lambda : \mathrm{G}_m \to T$ on the chosen Γ-orbit of $\Delta(B,T)$. □

3.0.5. Lemma Let H be a connected reductive group over an algebraically closed field, equipped with a pinning $(B, T, \{X_\alpha\})$. Let Γ be a finite group acting the pinned group $(H, B, T, \{X_\alpha\})$.

If P and Q are maximal proper Γ-stable parabolic subgroups of H, then one of the following is true:

- $P = Q$,
- $P \cap Q$ is a Levi of P, or

- there exists a Levi subgroup $M \subset P$ such that $M \cap Q$ is a maximal proper parabolic of M. Equivalently, the image of $P \cap Q$ in $P/U \cong M$ (where U is the unipotent radical of P) is a maximal proper parabolic of M.

Proof. By Lemma 3.0.1, there exists a maximal torus $T \subset P \cap Q$. There exists characters λ, $\mu : \mathrm{G}_m \to T$ such that $P = P_H(\lambda)$ and $Q = P_H(\lambda)$ (see, for example, [Sp98, Proposition 8.4.5]). Moreover, $M := Z_H(\lambda)$ is a Levi subgroup of P. We have $M \cap Q = M \cap P_G(\mu) = P_M(\mu)$ is a parabolic of M.

If $M \cap Q = M$, then $M \subset P \cap Q$ and thus by Lemma 3.0.4, we have either $P = Q$ or $P \cap Q = M$. □

4. A recursive classification of the irreducible components of $\mathcal{X}_{{}^LG,\mathrm{red}}$

Starting from this section, we assume G is a reductive group over F that splits over K (whereas in previous sections G is defined over K and splits over E), for consistency of notation with [L23].

4.1. Remarks on classical reductive groups

In this paper, we are interested in quasi-split forms of groups G whose Dynkin diagram is one of the following

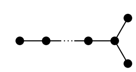
Type A		Type B	
Type C		Type D	

Table 4.1

We assume G is defined over a p-adic field F and splits over a tame extension K of F. We will exclude the triality, and assume $\mathrm{Gal}(K/F)$ acts either trivially, or acts as a reflection on $\mathrm{Dynkin}(G)$.

For simplicity, if D is the Dynkin diagram of G, we denote by $\mathrm{Res}_{K/F} D$ the Dynkin diagram of $\mathrm{Res}_{K/F} G$.

Let v be a vertex of $\mathrm{Dynkin}(G)$. After removing the $\mathrm{Gal}(K/F)$-orbit of v from $\mathrm{Dynkin}(G)$, we get the Dynkin diagram of a maximal proper Levi subgroup M of G. By inspecting Table 4.1, we conclude that if $\mathrm{Dynkin}(G) = X_n$, then

$$\mathrm{Dynkin}(M) = \begin{cases} \mathrm{Res}_{K/F} A_k \amalg X_{n-2-2k} & X = A \\ A_k \amalg X_{n-1-k} & X = B, C, \text{ or } D \end{cases}$$

for some integer k. We will denote such a Levi subgroup M by M_k and say it is *of niveau k*.

4.2. An axiomatized framework

In order to deal with classical groups including U_n, SO_n, Sp_{2n} and Spin_n simultaneously, we formulate a formal framework in this subsection.

4.2.1. Embedding into general linear group Although it is completely unnecessary for our arguments to work, we do opt to fix a natural embedding into a general linear group to *simplify our exposition*.

If we have an embedding into a general linear groups, we can freely use matrix notations throughout this section; if we don't, the same argument still works but we will have to use abstract and unintuitive notations which hurt the readability of the proofs.

4.2.2. Definition Let G be a connected reductive group over a p-adic field F which splits over a tame extension K/F.

Fix an action of $\mathrm{Gal}(K/F)$ on GL_N which fixes a pinning of GL_N and form a semidirect product $\mathrm{GL}_N \rtimes \mathrm{Gal}(K/F)$. Let $\iota_G : {}^L G \to \mathrm{GL}_N \rtimes \mathrm{Gal}(K/F)$ be an L-embedding.

A maximal proper Levi subgroup M of G is said to be *(ι_G-)classical of niveau k* if $M \cong \mathrm{Res}_{K/F}\mathrm{GL}_k \times H_M$ for some reductive group H_M, and for all parabolic ${}^L P \supset {}^L M$, we have, up to GL_N-conjugacy,

$$\iota_G({}^L P) \subset \begin{bmatrix} \mathrm{GL}_k & * & * \\ & \mathrm{GL}_{N-2k} & * \\ & & \mathrm{GL}_k \end{bmatrix} \rtimes \mathrm{Gal}(K/F),$$

$$\iota_G({}^L M) \subset \begin{bmatrix} \mathrm{GL}_k & & \\ & \mathrm{GL}_{N-2k} & \\ & & \mathrm{GL}_k \end{bmatrix} \rtimes \mathrm{Gal}(K/F),$$

$$\iota_G({}^L \mathrm{Res}_{K/F}\mathrm{GL}_d) \subset \begin{bmatrix} \mathrm{GL}_k & & \\ & I_{N-2k} & \\ & & \mathrm{GL}_k \end{bmatrix} \rtimes \mathrm{Gal}(K/F), \text{ and}$$

$$\iota_G({}^L H_M) \subset \begin{bmatrix} I_k & & \\ & \mathrm{GL}_{N-2k} & \\ & & *I_k \end{bmatrix} \rtimes \mathrm{Gal}(K/F).$$

Here $*I_k$ means all $k \times k$ non-zero scalar matrices.

We also say conjugates of ${}^L M$ are *classical of niveau k* in ${}^L G$.

4.2.3. Definition A *classical structure* of G is an L-embedding $\iota_G : {}^L G \to \mathrm{GL}_N \rtimes \mathrm{Gal}(K/F)$ such that

(CS1) all proper maximal Levi subgroups of ${}^L G$ are classical of niveau k for some k, and

for each parabolic L-parameter $\bar\rho : \mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$, there exists a maximal proper parabolic ${}^L P \subset {}^L G$ such that

(CS2) $\bar\rho$ is a Heisenberg type extension of some L-parameter $\bar\rho_M : \mathrm{Gal}_F \to {}^L M(\bar{\mathbb{F}}_p)$ ([L23C, Section 5]);

(CS3) the composition $\mathrm{Gal}_F \xrightarrow{\bar\rho_M} {}^L M \to {}^L \mathrm{Res}_{K/F}\mathrm{GL}_d$ is elliptic;

(CS4) if we endow $U(\bar{\mathbb{F}}_p) \subset \begin{bmatrix} I_k & \mathrm{Mat}_{k\times(N-2k)} & \mathrm{Mat}_{k\times k} \\ & I_{N-2k} & \mathrm{Mat}_{(N-2k)\times k} \\ & & I_k \end{bmatrix}$ with Gal_F-action through $\bar\rho_M$ composed with the adjoint action, then there exist isomorphisms

$$\alpha : H^\bullet(\mathrm{Gal}_K, \mathrm{Mat}_{k\times(N-2k)}(\bar{\mathbb{F}}_p)) \xrightarrow{\cong} H^\bullet(\mathrm{Gal}_K, \mathrm{Mat}_{(N-2k)\times k}(\bar{\mathbb{F}}_p))$$

$$\beta : H^\bullet(\mathrm{Gal}_K, \mathrm{Mat}_{(N-2k)\times k}(\bar{\mathbb{F}}_p)) \xrightarrow{\cong} H^\bullet(\mathrm{Gal}_K, \mathrm{Mat}_{k\times(N-2k)}(\bar{\mathbb{F}}_p))$$

such that $\alpha\circ\beta = \beta\circ\alpha = 1$, which induces a well-defined isomorphism

$$H^\bullet(\mathrm{Gal}_K, \mathrm{Mat}_{k\times(N-2k)}(\bar{\mathbb{F}}_p)) \xrightarrow{x\mapsto(x,\alpha x)} H^\bullet(\mathrm{Gal}_F, U/[U,U](\bar{\mathbb{F}}_p)),$$

and the symmetrized cup product (see, for example, [L23C, 3.11])

$$H^1(\mathrm{Gal}_F, U/[U,U](\bar{\mathbb{F}}_p)) \times H^1(\mathrm{Gal}_F, U/[U,U](\bar{\mathbb{F}}_p)) \to H^2(\mathrm{Gal}_F, [U,U](\bar{\mathbb{F}}_p))$$

is non-trivial unless $H^2(\mathrm{Gal}_F, [U,U](\bar{\mathbb{F}}_p)) = 0$,

(CS5) $\iota_{H_M} := \iota_G|_{H_M} : {}^L H_M \to \mathrm{GL}_{N-2k} \rtimes \mathrm{Gal}(K/F)$ is a classical structure of H_M.

We say a classical structure (G, ι_G) satisfies *Emerton-Gee's numerical criterion* if

(CS6) the locus

$$\{x | \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\mathrm{Gal}_K}(\bar{\alpha}, \iota_{H_M}(x)|_{\mathrm{Gal}_K}) \geq 1\}$$

in $\mathcal{X}_{F,{}^L H_M,\mathrm{red}}$ is either empty, or of dimension at most $[F:\mathbb{Q}_p]\dim \widehat{H}_M/B_{\widehat{H}_M} - 1$,

where

- $B_{\widehat{H}_M}$ is a Borel of $\widehat{H}_M$,
- $M = \mathrm{Res}_{K/F}\mathrm{GL}_k \times H_M$ is a ι_G-classical Levi subgroup of G.

(CS7) the locus

$$\{x | \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\mathrm{Gal}_K}(\bar{\alpha}, \iota_{H_M}(x)|_{\mathrm{Gal}_K}) \geq s\}$$

in $\mathcal{X}_{F,{}^L H_M,\mathrm{red}}$ is either empty, or of dimension at most $[F:\mathbb{Q}_p]\dim \widehat{H}_M/B_{\widehat{H}_M} - s - 1$ for $s > 1$, and

(CS8) the locus of

$$\{x | \iota_M(x) = \begin{bmatrix} \bar{\alpha} & & \\ & * & \\ & & \bar{\alpha}' \end{bmatrix} \rtimes *,\ H^2(\mathrm{Gal}_K, \bar{\alpha}\otimes\bar{\alpha}'^{\vee}) \neq 0\}$$

in $\mathcal{X}_{F,{}^L M,\mathrm{red}}$ has dimension at most $[F:\mathbb{Q}_p]\dim \widehat{H}_M/\widehat{B}_{H_M} - 1$,

(CS9) $\dim \mathcal{X}_{F,{}^L G,\mathrm{red}} \leq [F:\mathbb{Q}_p]\dim \widehat{G}/B_{\widehat{G}}$ where $B_{\widehat{G}}$ is a Borel of $\widehat{G}$.

(CS10) all L-parameters $\mathrm{Gal}_F \to {}^L G(\bar{\mathbb{F}}_p)$ admits a de Rham lift $\mathrm{Gal}_F \to {}^L G(\bar{\mathbb{Z}}_p)$ of regular Hodge-Tate type.

We remark that (CS9-CS10) follows from (CS1-CS8) by our previous work [L23C]. The proof of [L23C, Theorem 3] only makes formal use of (CS1-CS8). We list (CS9-CS10) in 4.2.3 for the sake of simplicity.

The following lemma is a generalization of the fact that if a matrix has two linearly independent eigenvectors, then it is conjugate to a matrix of the form $\begin{bmatrix} * & 0 & \dots \\ 0 & * & \dots \\ 0 & 0 & \dots \end{bmatrix}$.

4.2.4. Lemma Let G be a connected reductive group equipped with a classical structure. Let ${}^L P_1$ and ${}^L P_2$ be maximal proper parabolic subgroups of ${}^L G$ whose Levi factors ${}^L M_1$ and ${}^L M_2$ (resp.) are classical of niveau 1.

If ${}^L P_1 \neq {}^L P_2$ and the neutral component of ${}^L P_1 \cap {}^L P_2$ is not a reductive group, then there exists a maximal proper parabolic ${}^L Q \supset {}^L P_1 \cap {}^L P_2$ whose Levi ${}^L N = {}^L(\mathrm{Res}_{K/F}\mathrm{GL}_2 \times H_N)$ is classical of niveau 2 such that the image of

$${}^L P_1 \cap {}^L P_2 \hookrightarrow {}^L Q \to {}^L N \to {}^L \mathrm{Res}_{K/F}\mathrm{GL}_2$$

is contained in ${}^L \mathrm{Res}_{K/F} T$ where T is the maximal torus of GL_2.

Proof. By Lemma 3.0.5, the image of ${}^LP_1 \cap {}^LP_2 \to {}^LP_1/U_1 \cong {}^LM_1$, which we denote by ${}^LP_{12}$, is a proper parabolic of ${}^LM_1$. The neutral componenet of ${}^LP_{12}$ is not a torus. Since ${}^LM_1 \cong {}^L(\operatorname{Res}_{K/F}\mathrm{G}_m \times H_{M_1})$, the image of ${}^LP_{12}$ in ${}^LH_{M_1}$ is a proper parabolic, and is thus classical of niveau 1 since ${}^LP_2$ is.

We freely use the embedding ι from the definition of classicality below.

We have ${}^LP_1 = P_{{}^LG}(\lambda_1)$ and ${}^LP_2 = P_{{}^LG}(\lambda_2)$ where $\lambda_1 : t \mapsto \begin{bmatrix} t & & & & \\ & 1 & & & \\ & & I_{N-4} & & \\ & & & 1 & \\ & & & & t^{-1} \end{bmatrix}$ and $\lambda_2 : t \mapsto \begin{bmatrix} 1 & & & & \\ & t & & & \\ & & I_{N-4} & & \\ & & & t^{-1} & \\ & & & & 1 \end{bmatrix}$. Therefore ${}^LP_1 \cap {}^LP_2 = P_{{}^LG}(\lambda_1) \cap P_{{}^LG}(\lambda_2) \subset \begin{bmatrix} * & & \dots \\ & * & \dots \\ & & \dots \end{bmatrix} \rtimes *$. Set $\mu = t \mapsto \begin{bmatrix} tI_2 & & \\ & I_{N-4} & \\ & & t^{-1}I_2 \end{bmatrix}$ and ${}^LQ := P_{{}^LG}(\mu)$ and we are done. $\square$

4.2.5. Proposition If K is tamely ramified over $\mathbb{Q}_p$ and G admits a classical structure (we only need (CS9-CS10) of Definition 4.2.3), then $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is equidimensional of dimension $[F:\mathbb{Q}_p]\widehat{G}/B_{\widehat{G}}$.

Proof. By Definition 4.2.3(CS9), we have $\dim \mathcal{X}_{F,{}^LG,\mathrm{red}} \leq [F:\mathbb{Q}_p]\widehat{G}/B_{\widehat{G}}$.

Since $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is of finite type, finite type points are dense in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$. By [Stacks, Tag 0A21], it suffices to show the complete localizations of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ at all finite type points have dimension at least $[F:\mathbb{Q}_p]\widehat{G}/B_{\widehat{G}}$. By Definition 4.2.3(CS10), all finite type points have a de Rham lift of regular Hodge type; by Lemma 2.6.2, the local dimension of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ at a finite type point is at least the dimension of a de Rham Galois lifting ring of regular Hodge type $-\dim \widehat{G}$, which is equal to $[F:\mathbb{Q}_p]\widehat{G}/B_{\widehat{G}}$. $\square$

4.2.6. Lemma If $G = G_1 \times G_2$ is a product of quasi-split tame reductive groups over F, then

$$\mathcal{X}_{F,{}^LG} \cong \mathcal{X}_{F,{}^LG_1} \times \mathcal{X}_{F,{}^LG_2}.$$

Proof. Since we have L-homomorphisms ${}^LG \to {}^LG_i$ and ${}^LG_i \to {}^LG$ for $i = 1,2$. We can construct morphisms $\mathcal{X}_{F,{}^LG} \to \mathcal{X}_{F,{}^LG_1} \times \mathcal{X}_{F,{}^LG_2}$ and $\mathcal{X}_{F,{}^LG_1} \times \mathcal{X}_{F,{}^LG_2} \to \mathcal{X}_{F,{}^LG}$ which are clearly inverse to each other. $\square$

4.3. Some nowhere dense substacks of $\mathcal{X}_{{}^LG,\mathrm{red}}$

Assume K is tamely ramified over $\mathbb{Q}_p$ and G admits a classical structure which satisfies Emerton-Gee's numerical criterion.

When we speak of

> "the locus of … in the moduli stack of something",

we mean

> "the scheme-theoretic closure of the scheme-theoretic image of all families of something whose $\bar{\mathbb{F}}_p$-points are of the form … in the moduli stack of something".

So a "locus" is techinically always a closed substack. However, since we are interested in dimension analysis only, it is almost always harmless to replace a "locus" by its dense open substacks.

When we speak of "the moduli of L-parameters", we always mean "the moduli of (φ, Γ)-modules" in the sense of [L23B].

When we write $H^\bullet(\mathrm{Gal}_F, -)$, we always mean (φ, Γ)-cohomology (or the cohomology of the corresponding Herr complex).

4.3.1. Lemma Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of finite type algebraic stacks over $\operatorname{Spec} \bar{\mathbb{F}}_p$.

Let G_m act trivially on G_m and denote by $[\mathrm{G}_m / \mathrm{G}_m]$ the quotient stack. Let $S = \amalg[\mathrm{G}_m / \mathrm{G}_m]$ be the disjoint union of finitely many copies of $[\mathrm{G}_m / \mathrm{G}_m]$.

Let $\mathcal{X} \to S$ be a morphism. If for each $x : [\operatorname{Spec} \bar{\mathbb{F}}_p / \mathrm{G}_m] \hookrightarrow S$, the scheme-theoretic image of the fiber $\mathcal{X} \times_{S,x} [\operatorname{Spec} \bar{\mathbb{F}}_p / \mathrm{G}_m]$ in $\mathcal{Y}$ has dimension at most d, then the scheme-theoretic image of $\mathcal{X}$ in $\mathcal{Y}$ has dimension at most $(d + 1)$.

Proof. It is harmless to assume $\mathcal{X}$ is irreducible. Write $\mathcal{Z}$ for the scheme-theoretic image of $\mathcal{X}$ in $\mathcal{Y}$.

By [Stacks, 0DS4], we can replace $\mathcal{X}$ by a dense open such that $\dim \mathcal{X} - \dim \mathcal{Z} = \dim \mathcal{X}_{f(t)}$ for all $t \in |\mathcal{X}|$. The lemma follows from applying [Stacks, 0DS4] to $\mathcal{X} \to \mathcal{Y}$, $\mathcal{X} \to S$ and $\mathcal{X} \times_{S,x} [\operatorname{Spec} \bar{\mathbb{F}}_p / \mathrm{G}_m] \to \mathcal{Y}$. $\square$

4.3.2. Lemma Let ${}^L P$ be a classical maximal proper parabolic subgroup of ${}^L G$ of niveau k. Write

$$ {}^L M = {}^L(\operatorname{Res}_{K/F} \mathrm{GL}_k \times H_M) $$

for the Levi factor of ${}^L P$ and denote by U the unipotent radical of ${}^L P$.

Let $\operatorname{Spec} A$ be a reduced, irreducible, and finite type $\bar{\mathbb{F}}_p$-scheme, and let $\operatorname{Spec} A \to \mathcal{X}_{F, {}^L H_M, \mathrm{red}}$ be a basic morphism (see [L23B, 10.1] for the definition). Write $\mathcal{Z}_A$ for the scheme-theoretic image of $\operatorname{Spec} A$ in $\mathcal{X}_{F, {}^L H_M, \mathrm{red}}$.

Write $\mathcal{Z}_{A,G}$ for the locus L-parameters of the form

$$ \begin{bmatrix} \bar{\alpha} & * & * \\ & \bar{\tau} & * \\ & & \bar{\beta} \end{bmatrix} \rtimes * \tag{$\dagger$} $$

in $\mathcal{X}_{{}^L G, \mathrm{red}}$ where $\bar{\alpha}$ is an irreducible Gal_K-representation and $\bar{\tau} \rtimes *$ corresponds to a $\bar{\mathbb{F}}_p$-point of $\operatorname{Spec} A$.

If either $\mathcal{Z}_A$ is nowhere dense in $\mathcal{X}_{F, {}^L H_M, \mathrm{red}}$ or $k > 1$, then $\mathcal{Z}_{A,G}$ is nowhere dense in $\mathcal{X}_{{}^L G, \mathrm{red}}$.

Proof. **(Step 1)** We first consider the part of the locus where $H^2(\mathrm{Gal}_K, \bar{\alpha} \otimes \bar{\tau}^\vee) > 0$. By the local Tate duality, $\bar{\tau}$ determines $\bar{\alpha}$ and hence $\bar{\beta}$ up to finite ambiguity. We can thus divide $\mathcal{Z}_{A,G}$ into finitely many locally closed substacks where

- $\bar{\alpha}$ and $\bar{\beta}$ are uniquely determined by $\bar{\tau}$, and where
- either $\bar{\alpha}(1) = \bar{\beta}$, or $\bar{\alpha}(1) \neq \bar{\beta}$ holds (at all $\bar{\mathbb{F}}_p$-points).

Since we only care about the codimension, we can replace $\mathcal{Z}_{A,G}$ by one of the finitely many substacks described above. Recall that $\mathcal{X}_{F, {}^L M, \mathrm{red}} = \mathcal{X}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}} \times \mathcal{X}_{F, {}^L H, \mathrm{red}}$. The irreducible part $\mathcal{X}^{\mathrm{ell}}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}}$ of $\mathcal{X}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}}$ admits a coarse moduli space $X^{\mathrm{ell}}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}}$ (there are only finitely many irreducible mod p Galois representations up to unramified Galois characters, so $\mathcal{X}^{\mathrm{ell}}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}}$ is a disjoint union of finitely many copies of $[\mathrm{G}_m / \mathrm{G}_m]$ by Schur lemma) and the tautological map $\mathcal{X}^{\mathrm{ell}}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}} \to X^{\mathrm{ell}}_{F, {}^L \operatorname{Res}_{K/F} \mathrm{GL}_k, \mathrm{red}}$ admits a section (since we have natural maps

$[\mathrm{G}_m / \mathrm{G}_m] \to \mathrm{G}_m \to [\mathrm{G}_m / \mathrm{G}_m]$). Since $\bar{\alpha}$ and $\bar{\beta}$ are uniquely determined by $\bar{\tau}$, we have a morphism $\mathcal{Z}_{A,G} \to X^{\mathrm{ell}}_{F,{}^L\mathrm{Res}_{K/F}\mathrm{GL}_k,red}$ and thus we have a morphism $\operatorname{Spec} A \to \mathcal{Z}_{A,G} \to \mathcal{X}_{F,{}^LM,\mathrm{red}}$.

Write $(\operatorname{Spec} A)_s \subset \operatorname{Spec} A$ for the locally closed subscheme where

$$\dim H^2(\mathrm{Gal}_K, \bar{\alpha} \otimes \bar{\tau}^\vee) = s$$

is constant. Denote by $\mathcal{Z}_{A,M,s}$ for the scheme-theoretic image of $(\operatorname{Spec} A)_s$ in $\mathcal{X}_{F,{}^LM,\mathrm{red}}$, and denote by $\mathcal{Z}_{A,s}$ for the scheme-theoretic image of $(\operatorname{Spec} A)_s$ in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$. Since $\mathrm{Aut}(\bar{\alpha})$ is 1-dimensional, we have

$$\dim \mathcal{Z}_{A,M,s} = \dim \mathcal{Z}_{A,s} - 1.$$

By (a mild variant of) [L23C, Lemma 10.8], we have

$$(\bullet) \qquad \dim \mathcal{Z}_{A,G} \le \max_{s>0} \left(\dim \mathcal{Z}_{A,M,s} + s - c + [F:\mathbb{Q}_p] \dim U + \begin{cases} 1 & \bar{\alpha}(1) = \bar{\beta} \\ 0 & \bar{\alpha}(1) \neq \bar{\beta} \end{cases} \right)$$

where c is the codimension of the cup product vanishing locus (see loc. cit. for the precise definition). If $\bar{\alpha}(1) = \bar{\beta}$, then by the non-vanishing of cup products (Definition 4.2.3(CS4)), we have $c \ge 1$; so

$$-c + \begin{cases} 1 & \bar{\alpha}(1) = \bar{\beta} \\ 0 & \bar{\alpha}(1) \neq \bar{\beta} \end{cases} \le 0$$

holds under either circumstance. Thus,

$$(*) \qquad \dim \mathcal{Z}_{A,G} \le \max_{s>0} (\dim \mathcal{Z}_{A,s} - 1 + s + [F:\mathbb{Q}_p] \dim U)$$

Put

$$\begin{aligned} \operatorname{codim} \mathcal{Z}_{A,G} &:= \dim \mathcal{X}_{F,{}^LG,\mathrm{red}} - \dim \mathcal{Z}_{A,G} \\ \operatorname{codim} \mathcal{Z}_{A,s} &:= \dim \mathcal{X}_{F,{}^LH_M,\mathrm{red}} - \dim \mathcal{Z}_{A,s}. \end{aligned}$$

By Proposition 4.2.5, $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is equidimensional of dimension $[F:\mathbb{Q}_p]\widehat{G}/B_{\widehat{G}}$, and thus

$$\dim \mathcal{X}_{F,{}^LG,\mathrm{red}} = \dim \mathcal{X}_{F,{}^LH_M,\mathrm{red}} + [F:\mathbb{Q}_p]\frac{k(k-1)}{2} + [F:\mathbb{Q}_p] \dim U.$$

We can rewrite $(*)$ as

$$\operatorname{codim} \mathcal{Z}_{A,G} \ge \min_{s>0} \left(\operatorname{codim} \mathcal{Z}_{A,s} + 1 - s + [F:\mathbb{Q}_p]\frac{k(k-1)}{2} \right)$$

Put $c_s := \operatorname{codim} \mathcal{Z}_{A,s} + 1 - s + [F:\mathbb{Q}_p]\frac{k(k-1)}{2}$.

(Step 1-1) We first investigate the $s = 1$ case. We have $c_1 = \operatorname{codim} \mathcal{Z}_{A,s} + [F:\mathbb{Q}_p]\frac{k(k-1)}{2}$. It is clear that $c_1 > 0$ if either $\operatorname{codim} \mathcal{Z}_{A,s} > 0$ or $k > 1$.

(Step 1-2) Now suppose $k > 2$.

Recall that up to "unramified twists" (see [EG23] for the precise definition), there are only finitely many irreducible representations $\bar{\alpha}$ of Gal_K of rank k. As a consequence, the irreducible locus S of $\mathcal{X}_{K,\mathrm{GL}_k,\mathrm{red}}$ is a disjoint union of finitely many copies of $[\mathrm{G}_m / \mathrm{G}_m]$. Since in our setting, $\bar{\alpha}$ is uniquely determined by $\bar{\tau}$, there is a well-defined morphism $\operatorname{Spec} A \to S$ sending $\bar{\tau}$ to $\bar{\alpha}$. Fix an irreducible

Gal_K-representation $\bar{\alpha}$, which corresponds to a morphism $x : [\operatorname{Spec}\bar{\mathbb{F}}_p/\mathrm{G}_m] \hookrightarrow S$. The scheme-theoretic image of $\operatorname{Spec} A \times_{S,x} [\operatorname{Spec}\bar{\mathbb{F}}_p/\mathrm{G}_m]$ in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$ is contained in the scheme-theoretic closure of the locus

$$\{y|\dim_{\bar{\mathbb{F}}_p}\mathrm{Hom}_{\mathrm{Gal}_K}(\bar{\alpha}(-1),y)=s\}\subset\mathcal{X}_{F,{}^LH_M,\mathrm{red}},$$

and thus has dimension at most $\dim\mathcal{X}_{F,{}^LH_M,\mathrm{red}}-s-1$ by Definition 4.2.3(CS7). By Lemma 4.3.1, we have $\operatorname{codim}\mathcal{Z}_{A,s}\geq s$; and therefore $c_s=\operatorname{codim}\mathcal{Z}_{A,s}+1-s+[F:\mathbb{Q}_p]\frac{k(k-1)}{2}\geq 1+[F:\mathbb{Q}_p]\frac{k(k-1)}{2}>0$.

(Step 2) Now we consider the part of the locus where $\dim_{\bar{\mathbb{F}}_p}H^2(\mathrm{Gal}_K,\bar{\alpha}\otimes\bar{\tau}^\vee)=0$. For ease of the notation, replace $\mathcal{Z}_{A,G}$ by the substack where $H^2(\mathrm{Gal}_K,\bar{\alpha}\otimes\bar{\tau}^\vee)=0$.

The coarse moduli space of $[\mathrm{G}_m/\mathrm{G}_m]$ is G_m, and thus the coarse moduli space $\operatorname{Spec} B$ of the irreducible locus of $\mathcal{X}_{K,\mathrm{GL}_k,\mathrm{red}}$ is a disjoint union of finitely many copies of G_m. We thus have a morphism $\operatorname{Spec} B\to\mathcal{X}_{K,\mathrm{GL}_k,\mathrm{red}}$. Consider the composite

$$\operatorname{Spec} A\times\operatorname{Spec} B\to\mathcal{X}_{K,\mathrm{GL}_k,\mathrm{red}}\times\mathcal{X}_{F,{}^LH_M,\mathrm{red}}\cong\mathcal{X}_{F,\mathrm{Res}\,F/K\,\mathrm{GL}_k,\mathrm{red}}\times\mathcal{X}_{F,{}^LH_M,\mathrm{red}}\cong\mathcal{X}_{F,{}^LM,\mathrm{red}}$$

(see [L23B, Proposition 1] and Lemma 4.2.6). By [L23B, Lemma 10.1.1], there exists a scheme-theoretically surjective and finitely presented morphism $\coprod_i\operatorname{Spec} C_i\to\operatorname{Spec} A\otimes B$ such that each $\operatorname{Spec} C_i\to\mathcal{X}_{F,{}^LM,\mathrm{red}}$ is a U-basic morphism of a certain cohomology type. Note that $\mathcal{Z}_{A,G}$ is the scheme-theoretic union of the scheme-theoretic image of $\operatorname{Spec} C_i\times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}}\mathcal{X}_{F,{}^LP,\mathrm{red}}$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$.

It is harmless to replace $\mathcal{Z}_{A,G}$ by the scheme-theoretic image of

$$\mathcal{Y}_i:=\operatorname{Spec} C_i\times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}}\mathcal{X}_{F,{}^LP,\mathrm{red}}$$

in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$. Here, we have secretly replaced $\operatorname{Spec} C_i$ by a locally closed affine subscheme to ensure $\dim H^2(\mathrm{Gal}_K,\bar{\alpha}\otimes\bar{\tau}^\vee)=0$ over $\mathcal{Z}_{A,G}$. Note that there exists a morphism $\mathcal{Y}_i\to\operatorname{Spec} C_i\to\operatorname{Spec} B$.

Fix a point $x:\operatorname{Spec}\bar{\mathbb{F}}_p\to\operatorname{Spec} B$ and denote by $\mathcal{Y}_{i,x}$ the fiber $\mathcal{Y}_i\times_{\operatorname{Spec} B,x}\operatorname{Spec}\bar{\mathbb{F}}_p$. Denote by $\mathcal{Z}_{A,G,x}$ the scheme-theoretic image of $\mathcal{Y}_{i,x}$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$; denote by $\mathcal{Z}_{A,M,x}$ the scheme-theoretic image of $\operatorname{Spec} C_i\times_{\operatorname{Spec} B,x}\operatorname{Spec}\bar{\mathbb{F}}_p$ in $\mathcal{X}_{F,{}^LM,\mathrm{red}}$; and denote by $\mathcal{Z}_{A,x}$ the scheme-theoretic image of $\operatorname{Spec} C_i\times_{\operatorname{Spec} B,x}\operatorname{Spec}\bar{\mathbb{F}}_p$ in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$.

By possibly replacing $\operatorname{Spec} C_i$ by the inverse image of an irreducible component of $\mathcal{Z}_A$, we can assume $\mathcal{Z}_A$ is irreducible.

There are two possibilities: $\mathcal{Z}_{A,x}=\mathcal{Z}_A$ and $\mathcal{Z}_{A,x}\neq\mathcal{Z}_A$.

(Step 2-1) We first consider the case where $\mathcal{Z}_{A,x}=\mathcal{Z}_A$. Now $\bar{\alpha}$ is constant over $\mathcal{Z}_{A,G}$ and the proof in (Step 1) works verbatim.

(Step 2-2) Finally, we consider the case where $\mathcal{Z}_{A,x}\neq\mathcal{Z}_A$. The irreducibility of $\mathcal{Z}_A$ implies $\dim\mathcal{Z}_{A,x}\leq\dim\mathcal{Z}_A-1$.

By (a mild variant of) [L23C, Lemma 10.8], we have

$$\begin{aligned}\dim\mathcal{Z}_{A,G,x}&\leq\left(\dim\mathcal{Z}_{A,M,x}-c+[F:\mathbb{Q}_p]\dim U+\begin{cases}1 & \bar{\alpha}(1)=\bar{\beta}\\ 0 & \bar{\alpha}(1)\neq\bar{\beta}\end{cases}\right)\\&\leq\dim\mathcal{Z}_{A,M,x}+[F:\mathbb{Q}_p]\dim U\\&=\dim\mathcal{Z}_{A,x}-1+[F:\mathbb{Q}_p]\dim U\\&\leq\dim\mathcal{Z}_A-2+[F:\mathbb{Q}_p]\dim U.\end{aligned}$$

By Lemma 4.3.1, we have

$$\dim\mathcal{Z}_{A,G}\leq\max_x\dim\mathcal{Z}_{A,G,x}+1\leq(\dim\mathcal{Z}_A-1)+[F:\mathbb{Q}_p]\dim U.$$

The proof in (Step 1-2) shows

$$\mathrm{codim}\mathcal{Z}_{A,G} := \dim \mathcal{X}_{F,{}^LG,\mathrm{red}} - \dim \mathcal{Z}_{A,G} \geq (\mathrm{codim}\mathcal{Z}_A - 1) + 1 + [F:\mathbb{Q}_p]\frac{k(k-1)}{2},$$

and thus either $\mathcal{Z}_A$ being nowhere dense or $k > 1$ implies $\mathcal{Z}_{A,G}$ is nowhere dense. $\square$

A Borel of a disconnected reductive group H is defined to be a minimal element of the set of all *big pseudo-parabolics* (see [L23, Definition 2.2.5]). We warn the reader that in the literature, some authors define Borels of H to be Borels of the neutral component H°.

4.3.3. Lemma Assume G is not a torus.

The locus of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ consisting of L-parameters $\mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$ that does not factor through any Borel is nowhere dense.

Proof. If a mod p L-parameter does not factor through any Borel, then it is either elliptic or factors through a parabolic ${}^LP_k$ whose Levi is of the form ${}^L(\operatorname{Res}_{K/F}\mathrm{GL}_d \times H_M)$, $k > 1$. By [L23, Theorem B] and Corollary 2.8.2, the elliptic locus has dimension at most 0; by Lemma 4.3.2, the locus of L-parameters that factors through ${}^LP_k$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is nowhere dense. $\square$

By Lemma 4.3.3, to understand the irreducible components of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$, it suffices to understand the locus of L-parameters that factors through a Borel.

4.4. The parabolic Emerton-Gee stacks: the Borel case

Let $(\widehat{B},\widehat{T})$ be a $\mathrm{Gal}(K/F)$-stable Borel pair of $\widehat{G}$, which exists by Corollary 3.0.2. Put ${}^LB = \widehat{B} \rtimes \mathrm{Gal}(K/F)$ and ${}^LT = \widehat{T} \rtimes \mathrm{Gal}(K/F)$. Write U for the unipotent radical of $\widehat{B}$.

Let A be a reduced finite type $\bar{\mathbb{F}}_p$-algebra. Let $\operatorname{Spec} A \to \mathcal{X}_{F,{}^LT,\mathrm{red}}$ be a U-basic morphism. The stack $\operatorname{Spec} A \underset{\mathcal{X}_{F,{}^LT,\mathrm{red}}}{\times} \mathcal{X}_{F,{}^LB}$ is algebraic and finitely presented over $\operatorname{Spec}\bar{\mathbb{F}}_p$ ([L23B, Proposition 10.1.8]).

Let $\widehat{B}'$ be another $\mathrm{Gal}(K/F)$-stable Borel and put ${}^LB' = \widehat{B}' \rtimes \mathrm{Gal}(K/F)$. By Corollary 3.0.2, there exists a $\mathrm{Gal}(K/F)$-stable maximal torus of $\widehat{G}$ in $\widehat{B} \cap \widehat{B}'$. Say $\widehat{T} \subset \widehat{B} \cap \widehat{B}'$ is $\mathrm{Gal}(K/F)$-stable. Put $V = U \cap \widehat{B}'$.

4.4.1. Lemma We have ${}^LB \cap {}^LB' \cong V \rtimes {}^LT$.

As a consequence, $\operatorname{Spec} A \underset{\mathcal{X}_{F,{}^LT,\mathrm{red}}}{\times} \mathcal{X}_{F,{}^LB\cap{}^LB'}$ is a finite type algebraic stack over $\bar{\mathbb{F}}_p$.

Proof. It is clear that V is stable under the conjugation action of ${}^LT$. Since U is the kernel of the quotient map ${}^LB \to {}^LT$, V is the kernel of the quotient map ${}^LB \cap {}^LB' \to {}^LT$.

Note that [L23B, Section 10.1] applies to all groups that are semidirect product of a nilpotent algebraic group and an L-group. In particular, [L23B, Proposition 10.1.8] implies $\operatorname{Spec} A \underset{\mathcal{X}_{F,{}^LT,\mathrm{red}}}{\times} \mathcal{X}_{F,{}^LB\cap{}^LB'}$ is a finite type algebraic stack over $\bar{\mathbb{F}}_p$. $\square$

4.4.2. Lemma There is a stratification $\{\mathcal{X}_i\}_{i\in I}$ by finitely many locally closed substacks on $\mathcal{X}_{F,{}^LT,\mathrm{red}}$ such that for any reduced finite type scheme $\operatorname{Spec} A$ and any morphism $\operatorname{Spec} A \to \mathcal{X}_{F,{}^LT,\mathrm{red}}$, the base change

$$\operatorname{Spec} A \times_{\mathcal{X}_{F,{}^LT,\mathrm{red}}} \mathcal{X}_i \to \mathcal{X}_{F,{}^LT,\mathrm{red}}$$

is a U-basic morphism. Moreover, we can ensure if $i \neq j$ then $\mathcal{X}_i$ and $\mathcal{X}_j$ have different cohomology types (in the sense of [L23B, Section 8]).

Proof. Note that $\operatorname{Lie} U = \prod_{\alpha \in R^+(\widehat{B},\widehat{T})} U_\alpha$. For each mod p L-parameter $x : \operatorname{Spec} \bar{\mathbb{F}}_p \to \mathcal{X}_{F,{}^LT}$, we have $H^0(\mathrm{Gal}_F, U_\alpha(\bar{\mathbb{F}}_p))$ = either 0 or $\bar{\mathbb{F}}_p$ depending whether x acts trivially on U_α; similarly, $H^2(\mathrm{Gal}_F, U_\alpha(\bar{\mathbb{F}}_p))$ = either 0 or $\bar{\mathbb{F}}_p$ depending whether x acts by the cyclotomic character on U_α; by the local Euler characteristic, $\dim H^1(\mathrm{Gal}_F, U_\alpha(\bar{\mathbb{F}}_p))$ is determined by H^0 and H^2.

A morphism $\operatorname{Spec} B \to \mathcal{X}_{F,{}^LT,\mathrm{red}}$ is U-basic if $\dim_{\bar{\mathbb{F}}_p} H^\bullet(\mathrm{Gal}_F, U_\alpha(\bar{\mathbb{F}}_p))$ is constant on $\operatorname{Spec} B$ for all $\alpha \in R^+(\widehat{B},\widehat{T})$.

The condition that x acts on $U_\alpha(\bar{\mathbb{F}}_p)$ trivially (or by the cyclotomic character) is a closed condition on the moduli stack. Thus we can divide $\mathcal{X}_{F,{}^LT,\mathrm{red}}$ into $3^{R^+(\widehat{B},\widehat{T})}$ locally closed substacks satisfying the required condition. $\square$

4.4.3. Definition By descent,

$$\mathcal{X}_{F,{}^LB,i} := \mathcal{X}_i \times_{\mathcal{X}_{F,{}^LT,\mathrm{red}}} \mathcal{X}_{F,{}^LB}$$

and

$$\mathcal{X}_{F,{}^LB\cap{}^LB',i} := \mathcal{X}_i \times_{\mathcal{X}_{F,{}^LT,\mathrm{red}}} \mathcal{X}_{F,{}^LB\cap{}^LB'}$$

are both finite type algebraic stacks over $\bar{\mathbb{F}}_p$.

By the Bruhat decomposition, up to $\widehat{B}$-conjugacy, there are only finitely many Borels ${}^LB'$. The union of the scheme-theoretic image of $\mathcal{X}_{F,{}^LB\cap{}^LB',i}$ in $\mathcal{X}_{F,{}^LB,i}$ for various $\widehat{B}'$ is thus a finite union of closed substacks, and we call it the *non-maximally non-split locus* and denote it by $\mathcal{X}^{\mathrm{nmns}}_{F,{}^LB,i}$.

The complement of the non-maximally non-split locus is by definition *the open substack of maximally non-split L-parameters*, which we denote by $\mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i}$.

4.4.4. Lemma The natural morphism $f : \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i} \to \mathcal{X}_{F,{}^LG}$ is a monomorphism.

Proof. By the geometric Shapiro's lemma [L23B, Proposition 1], it is harmless to assume $F = \mathbb{Q}_p$. Let A be a reduced finite type scheme over $\bar{\mathbb{F}}_p$ and let $x_A : \operatorname{Spec} A \to \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i}$ be a morphism.

Write $f_*(x_A)$ for the composition $\operatorname{Spec} A \to \mathcal{X}_{F,{}^LG}$.

By [Stacks, Tag 04ZZ], we want to show f is fully faithful. Since we are working with groupoids, it suffices to look at automorphisms. Note that automorphisms in either groupoids are defined to be automorphisms of the corresponding (φ,Γ)-module with ${}^LG$- or ${}^LB$-structure.

We claim that $\mathrm{Aut}_{\mathcal{X}_{F,{}^LG}}(f_*(x_A)) = \mathrm{Aut}_{\mathcal{X}_{F,{}^LB}}(x_A) \subset \widehat{B}(\mathbb{A}_{\mathbb{Q}_p,A})$. Assume the contraposition, and suppose $g_A \in \widehat{G}(\mathbb{A}_{\mathbb{Q}_p,A})$ is an automorphism of $f_*(x_A)$ and $g \notin \widehat{B}(\mathbb{A}_{\mathbb{Q}_p,A})$. We specialize at an $\bar{\mathbb{F}}_p$-point $\operatorname{Spec} \bar{\mathbb{F}}_p \to \operatorname{Spec} A$ such that the specialization (of g_A) $g_{\bar{\mathbb{F}}_p} \notin \widehat{B}(\mathbb{A}_{\mathbb{Q}_p,\bar{\mathbb{F}}_p})$; and let $x_{\bar{\mathbb{F}}_p} : \operatorname{Spec} \bar{\mathbb{F}}_p \to \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i}$ be the corresponding (φ,Γ)-module with ${}^LB$-structure with $\bar{\mathbb{F}}_p$-coefficients.

By the relation of (φ,Γ)-modules with Galois representations and Tannakian formalism, $f_*(x_{\bar{\mathbb{F}}_p})$ corresponds to an L-parameter $\bar{\rho}_{f_*(x_{\bar{\mathbb{F}}_p})} : \mathrm{Gal}_F \to {}^LG(\bar{\mathbb{F}}_p)$ and the automorphism $g_{\bar{\mathbb{F}}_p} \notin \widehat{B}(\mathbb{A}_{\mathbb{Q}_p,\bar{\mathbb{F}}_p})$ corresponds to an automorphism $g_{\bar{\rho}} \in \widehat{G}(\bar{\mathbb{F}}_p) - \widehat{B}(\bar{\mathbb{F}}_p)$ of $\bar{\rho}_{f_*(x_{\bar{\mathbb{F}}_p})}$.

By the construction of $f_*(x_{\bar{\mathbb{F}}_p})$, $\bar{\rho}_{f_*(x_{\bar{\mathbb{F}}_p})}$ factors through ${}^LB(\bar{\mathbb{F}}_p)$. Since $g_{\bar{\rho}}\bar{\rho}_{f_*(x_{\bar{\mathbb{F}}_p})}g_{\bar{\rho}}^{-1} = \bar{\rho}_{f_*(x_{\bar{\mathbb{F}}_p})}$, $\bar{\rho}_{f_*(x_{\bar{\mathbb{F}}_p})}$ also factors through $g_{\bar{\rho}}{}^LB(\bar{\mathbb{F}}_p)g_{\bar{\rho}}^{-1}$. Since $\widehat{B}$ is self-normalizing and $g_{\bar{\rho}} \notin \widehat{B}(\bar{\mathbb{F}}_p)$, we know that $\bar\rho_{f_*(x_{\bar{\mathbb{F}}_p})}$ factors through two distinct Borels of ${}^LG$. However, by the construction of $\mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i}$, all $\bar{\mathbb{F}}_p$-points should correspond to an L-parameter that factors through a unique Borel, and we get a cotradiction. □

4.4.5. Lemma Assume G is not a torus.

The (finite) union of the scheme-theoretic image of $\mathcal{X}_{F,{}^LB,i,\mathrm{red}}$ for all $i \in I$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is the whole $\mathcal{X}_{F,{}^LG,\mathrm{red}}$.

Proof. It is a restatement of Lemma 4.3.3. □

4.4.6. Lemma Assume G is not a torus. Let ${}^LB \neq {}^LB'$ be Borels of ${}^LG$.

(1) The scheme-theoretic image of $\mathcal{X}_{F,{}^LB\cap{}^LB',i,\mathrm{red}}$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is nowhere dense.

(2) The (finite) union of the scheme-theoretic image of $\mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i,\mathrm{red}}$ for all $i \in I$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is the whole $\mathcal{X}_{F,{}^LG,\mathrm{red}}$.

(3) We have $\dim \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i,\mathrm{red}} \leq \dim \mathcal{X}_{F,{}^LG,\mathrm{red}}$ for all $i \in I$.

Proof. By Lemma 4.4.5, (1) is equivalent to (2). By [Stacks, Tag 0DS4], if $\mathcal{X} \to \mathcal{Y}$ is a monomorphism, $\dim \mathcal{X} \leq \dim \mathcal{Y}$, and thus by Lemma 4.4.4, (2) implies (3).

By Lemma 4.2.4, ${}^LB \cap {}^LB' = \begin{bmatrix} * & 0 & \dots & * & * \\ 0 & * & \dots & * & * \\ & & [*] & \dots & \dots \\ & & & * & 0 \\ & & & & * \end{bmatrix} \rtimes *$, and thus L-parameters that correspond to $\bar{\mathbb{F}}_p$-points of $\mathcal{X}_{F,{}^LB\cap{}^LB',i}$ are of the form

$$\bar\rho = \begin{bmatrix} \bar\alpha & 0 & \dots & * & * \\ 0 & \bar\beta & \dots & * & * \\ & & [\bar\tau] & \dots & \dots \\ & & & \bar\beta' & 0 \\ & & & & \bar\alpha' \end{bmatrix} \rtimes *$$

We prove (1) by induction on the rank of B. The base case of the induction is the elliptic case which has been considered in the proof of Lemma 4.3.3. By Lemma 4.3.2, the locus of $\bar\rho$ where $\bar\tau$ is not maximally non-split form a nowhere dense substack. Thus, we assume $\bar\tau$ is maximally non-split in the rest of the proof.

(Step 1) The part where $\bar\alpha = \bar\beta$. If $\bar\alpha = \bar\beta$, then $\dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\mathrm{Gal}_K}(\bar\alpha, \bar\rho) \geq 2$ and by Definition 4.2.3(CS7), the part where $\bar\alpha = \bar\beta$ has nowhere dense scheme-theoretic image in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$.

Now we assume $\bar\alpha \neq \bar\beta$ in the rest of the proof. In particular, $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\tau^\vee)$ and $H^2(\mathrm{Gal}_K, \bar\beta \otimes \bar\tau^\vee)$ cannot be simultaneously non-trivial since $\bar\tau$ is maximally non-split in the sense that it factors through a unique Borel.

(Step 2) The part where $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\tau^\vee) \neq 0$.

Note that $\bar\rho$ factors through two Borels ${}^LB$ and ${}^LB'$ and if we interchange ${}^LB$ and ${}^LB'$, the two characters $\bar\alpha$ and $\bar\beta$ are interchanged. Since $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\tau^\vee)$ and $H^2(\mathrm{Gal}_K, \bar\beta \otimes \bar\tau^\vee)$ cannot be simultaneously non-trivial, we can skip (Step 2) and proceed to (Step 3), by possibly swapping ${}^LB$ and ${}^LB'$.

(Step 3) The part where $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\tau^\vee) = 0$.

(Step 3-1) The part where $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\alpha'^\vee) \neq 0$. Let ${}^LP$ be the maximal proper parabolic of ${}^LG$ containing ${}^LB$ that is classical of niveau 1, and denote by ${}^LM$ the Levi factor of ${}^LP$. Recall that $M = \mathrm{Res}_{K/F}\,\mathrm{G}_m \times H_M$. The locus of

$$\bar\mu = \begin{bmatrix} \bar\beta & \dots & * \\ & [\bar\tau] & \dots \\ & & \bar\beta' \end{bmatrix} \rtimes *$$

in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$ such that there are infinitely many $\bar\alpha$ such that

$$\bar\rho = \begin{bmatrix} \bar\alpha & & \\ & \bar\mu & \\ & & \bar\alpha' \end{bmatrix} \rtimes * = \begin{bmatrix} \bar\alpha & 0 & \dots & 0 & 0 \\ 0 & \bar\beta & \dots & * & 0 \\ & & [\bar\tau] & \dots & \dots \\ & & & \bar\beta' & 0 \\ & & & & \bar\alpha' \end{bmatrix} \rtimes *$$

satisfies $H^2(\mathrm{Gal}_K, \bar\alpha\otimes\bar\alpha'^\vee) \neq 0$ is nowhere dense in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$; for if otherwise, it contradicts Definition 4.2.3(CS8). By Lemma 4.3.2, we can assume for each $\bar\mu$ there are only finitely many $\bar\alpha$ such that $H^2(\mathrm{Gal}_K, \bar\alpha\otimes\bar\alpha'^\vee) \neq 0$. The finitude of the choice of $\bar\alpha$ allows us to assume $\bar\alpha$ is completely determined by $\bar\mu$ in the discussion below (by splitting this part into finitely many subparts). Running the arguments presented in Lemma 4.3.2 once again shows the part considered in (Step 3-1) is nowhere dense. To elaborate, fixing $\bar\alpha$ induces one codimension. The vanishing of the entry above $\bar\beta$ and to the right of $\bar\alpha$ induces $[K:F]$ codimensions. The non-vanishing of $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\alpha'^\vee)$ reduces the codimension by 1. Since $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\tau^\vee) = 0$, the codimension is not further reduced. To sum up, the codimension is at least $(1 + [K:F] - 1) > 0$. See the proof of Lemma 4.3.2 for the rigorous argument.

(Step 3-2) The part where $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\alpha'^\vee) = 0$. We continue the discussion at the end of (Step 3-1). Now $\bar\alpha$ is not fixed. But all H^2 vanish. The vanishing of the entry above $\bar\beta$ and to the right of $\bar\alpha$ induces $[K:F]$ codimensions. So this part has at least $[K:F]$ codimensions. ☐

4.5. The irreducible components of $\mathcal{X}_{{}^LG,\mathrm{red}}$

4.5.1. Lemma Let X and Y be algebraic stacks that are finitely presented over $\bar{\mathbb{F}}_p$. Let $f : X \to Y$ be a scheme-theoretically dominant morphism.

If Y is irreducible and all fibers of f are irreducible and have the same dimension d (see Definition [Stacks, Tag 0DRG]), then $\dim X \le \dim Y + d$ and there exists at most one irreducible component $C \subset X$ such that $\dim C = \dim X$; the equality $\dim X = \dim Y + d$ holds if and only if such an irreducible component C exists.

Proof. The inequality follows from [Stacks, 0DS4]. Assume there are two such irreducible components C_1 and C_2. Write Y_1 for the scheme-theoretic image of C_1 in Y. We have $\dim Y_1 \ge \dim C_1 - d$ by [Stacks, 0DS4]. Thus $\dim Y_1 \ge \dim Y$ and since Y is irreducible, $Y_1 = Y$. Similarly, if Y_2 is the scheme-theoretic image of C_2 in Y, then $Y_2 = Y$.

To show $C_1 = C_2$, by descent, it suffices to show $C_1 = C_2$ after a smooth base change by a smooth cover $Y' \to Y$. In particular, we can assume Y is a scheme. Note that both $f(C_1)$ and $f(C_2)$ are dense constructible subsets of Y. Since X and Y are Noetherian, the constructibility implies $f(C_1)$ (and $f(C_2)$, resp.) contains a dense open subset V_1 (and V_2, resp.) of Y. Set $V = V_1 \cap V_2$. $f^{-1}(V) \cap C_1$ is a dense open of C_1 and $f^{-1}(V) \cap C_2$ is a dense open of C_2. Since $\dim f^{-1}(V) \cap C_1 = \dim f^{-1}(V) \cap C_2$, we conclude $f^{-1}(V) \cap C_1 = f^{-1}(V) \cap C_2$ by applying [Stacks, 0DS4] (fibers of $f^{-1}(V) \cap C_i \to V$ are generically irreducible of dimension d on the source and therefore generically equal; note that both $f^{-1}(V) \cap C_i \to V$, $i = 1, 2$ are closed in $f^{-1}(V)$). Therefore, $C_1 = C_2$. □

4.5.2. Lemma Let X be a scheme of finite type over $\bar{\mathbb{F}}_p$. Let G be a smooth group scheme over $\bar{\mathbb{F}}_p$ that acts *trivially* on X.

If $[X/G]$ is equidimensional, then X is irreducible if and only if the quotient stack $[X/G]$ is irreducible.

Proof. Since G acts on X trivially, the coarse moduli sheaf of $[X/G]$ is representable by X. Thus there exists a forgetful morphism $f : [X/G] \to X$, which has a section, namely, the quotient map $X \to [X/G]$; as a consequence, if C_1 and C_2 are two distinct irreducible components of X, then their images in $[X/G]$ are also two distinct irreducible components.

Conversely, assume X is irreducible. Note that all fibers of $f : [X/G] \to X$ are of the form $[\operatorname{Spec} \kappa/G]$ and are irreducible of dimension $-\dim G$, where κ is a residue field of X. By Lemma 4.5.1, there exists at most one irreducible component of $[X/G]$ of dimension $\dim X - \dim G$. So it remains to show such an irreducible exists. Since X is of finite type, $[X/G]$ has only finitely many irreducible components; as a consequence, there exists an irreducible component of $\mathcal{C}$ of $[X/G]$ such that $\mathcal{C} \to X$ is scheme-theoretically dominant. Since X is a Noetherian scheme, $f(\mathcal{C})$ is a dense constructible subset of X, and thus contains a dense open U of X. By [Stacks, Tag 0DS4], $\dim \mathcal{C} \geq \dim f^{-1}(U) = \dim X - \dim G$. □

4.5.3. Lemma If $f : X \to Y$ is a scheme-theoretically dominant monomorphism between finite type equidimensional algebraic stacks over $\bar{\mathbb{F}}_p$, then X is irreducible if and only if Y is irreducible.

Proof. Suppose X is irreducible. Let C_i $(i \in I)$ be the finitely many irreducible components of Y. Since $f^{-1}(C_i)$ are all closed substacks of X and X is irreducible, there exists an $i_0 \in I$ such that $f^{-1}(C_{i_0}) = X$. Since f is scheme-theoretically dominant, $C_{i_0} = Y$ is irreducible.

Suppose Y is irreducible. Since f is a scheme-theoretically dominant monomorphism, $\dim X = \dim Y$ (by [Stacks, 0DS4]). Let C_1 and C_2 be two irreducible components of X, and let Z_1 and Z_2 be the scheme-theoretic image of C_1 and C_2, resp.. By [Stacks, 0DS4] again, $\dim C_1 = \dim Z_1 = \dim C_2 = \dim Z_2 = \dim X = \dim Y$. Since Y is irreducible, $Z_1 = Z_2$. By Chevalley's theorem on contructibility, $C_1 = C_2$, since $f(C_1)$ and $f(C_2)$ contains a common dense open of Y. □

4.5.4. Lemma Assume $p \neq 2$. Let $(-,-)$ be a symmetric bilinear pairing on the vector space $\bar{\mathbb{F}}_p^{\oplus N}$. A vector of $\bar{\mathbb{F}}_p^{\oplus N}$ can be represented by a tuple $\mathbf{x} = (x_1, \ldots, x_N)$. The function $f(\mathbf{x}) := (\mathbf{x}, \mathbf{x})$ is a homogeneous polynomial of degree 2 in N variables.

Let $X = \operatorname{Spec} \bar{\mathbb{F}}_p[x_1, \ldots, x_N]/(f(\mathbf{x}))$. If X is not an irreducible scheme, then the kernel of the pairing $(-,-)$ has dimension at least $(N-2)$.

Proof. Since $\operatorname{Spec} \bar{\mathbb{F}}_p[x_1, \ldots, x_N]$ is a PID, we can write $f = gh$ where g and h are homogeneous polynomials of degree 1 if X is not irreducible. We can regard g and h as elements of the dual vector space of $\bar{\mathbb{F}}_p^{\oplus N}$. Equip $\bar{\mathbb{F}}_p^{\oplus N}$ with the standard inner product and identify $\bar{\mathbb{F}}_p^{\oplus N}$ with its dual. We have

$$2(\mathbf{x}, \mathbf{x}') = f(\mathbf{x} + \mathbf{x}') - f(\mathbf{x}) - f(\mathbf{x}') = \mathbf{x}^t(gh^t + hg^t)\mathbf{x}.$$

Therefore if $\mathbf{x}$ is orthogonal to both h and g, then $\mathbf{x}$ lies in the kernel of $(-,-)$. We have $\dim\langle g, h\rangle^\perp \geq N - 2$. □

In applications, we always have $N > 2$ and $(-,-)$ is non-degenerate and thus the affine scheme X in Lemma 4.5.4 will be irreducible.

4.5.5. Lemma A dense subset of an irreducible topological space is irreducible.

Proof. Suppose X is irreducible and $Y \subset X$ is dense. Suppose $Y = Y_1 \cup Y_2$ where both Y_1 and Y_2 are closed in Y. Write $\mathrm{cl}(-)$ for the closure in X. Since $\mathrm{cl}(Y_1) \cup \mathrm{cl}(Y_2)$ is a closed subset of X containing Y, we have $X = \mathrm{cl}(Y_1) \cup \mathrm{cl}(Y_2)$. Thus either $\mathrm{cl}(Y_1) \subset \mathrm{cl}(Y_2)$ or $\mathrm{cl}(Y_1) \supset \mathrm{cl}(Y_2)$, and therefore either $Y_1 \subset Y_2$ or $Y_1 \supset Y_2$ since Y_1 and Y_2 are closed in Y. ☐

Let ${}^L B$ be a Borel of ${}^L G$.

4.5.6. Lemma The morphism

$$\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}} \to \mathcal{X}_{F,{}^L G,\mathrm{red}}$$

induces a bijection between irreducible components of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}}$ of maximal dimension and irreducible components of $\mathcal{X}_{F,{}^L G,\mathrm{red}}$.

Proof. The morphism $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}} \to \mathcal{X}_{F,{}^L G,\mathrm{red}}$ is scheme-theoretically dominant by Lemma 4.4.6(2), and is a monomorphism by Lemma 4.4.4.

It is harmless to replace $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}}$ by the scheme-theoretic union of its irreducible components of maximal dimension. By Proposition 4.2.5, Lemma 4.5.3 implies this lemma. ☐

4.5.7. Relatively Steinberg components Let ${}^L P$ be the maximal proper parabolic of ${}^L G$ of niveau 1 with Levi subgroup ${}^L M$ and unipotent radical U. Recall that $M \cong \mathrm{Res}_{K/F}\,\mathrm{G}_m \times H_M$.

The group homomorphisms

$$\begin{tikzcd} {}^L B \arrow[r] & {}^L P \arrow[r] \arrow[d] & {}^L G \\ & {}^L M & \end{tikzcd}$$

induces morphisms of stacks

$$\begin{tikzcd} \coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}} \arrow[r] & \mathcal{X}_{F,{}^L P} \arrow[r] \arrow[d] & \mathcal{X}_{F,{}^L G} \\ & \mathcal{X}_{F,{}^L M} & \end{tikzcd}$$

By Lemma 4.5.6, irreducible components of $\mathcal{X}_{F,{}^L G,\mathrm{red}}$ are identified with irreducible components of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}}$ of maximal dimension. We say an irreducible component of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^L B,i,\mathrm{red}}$ of maximal dimension is *relatively Steinberg* if its scheme-theoretic image in $\mathcal{X}_{F,{}^L M,\mathrm{red}}$ is *not* an irreducible component of $\mathcal{X}_{F,{}^L M,\mathrm{red}}$.

The main theorem of this paper is the following.

4.5.8. Theorem Assume either

- H_M is not a torus or
- there exists a surjection $\mathrm{Res}_{K/F}\,\mathrm{G}_m \to H_M$ (for example, if $H_M \cong \mathrm{Res}_{K/F}\,\mathrm{G}_m$ or U_1).

Then the following are true.

(1) If $\dim U/[U,U] \geq 2$, then there exists a natural bijection between the irreducible components of $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$ and the relatively Steinberg irreducible components of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$.

(2) There exists a natural bijection between the irreducible components of $\mathcal{X}_{F,{}^LM,\mathrm{red}}$ and the relatively non-Steinberg irreducible components of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$.

4.5.9. Lemma If C is an irreducible component of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i,\mathrm{red}}$ of maximal dimension, then its scheme-theoretic image in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$ is an irreducible component.

Proof. Let $\operatorname{Spec} A \to C$ be a scheme-theoretically surjective, finite type morphism which is U-basic (see [L23B, Lemma 10.1.1] for the existence of $\operatorname{Spec} A$). Suppose the scheme-theoretic image Z of C in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$ is not an irreducible component. Since $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$ is equidimensional, Z is nowhere dense in $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$. By Lemma 4.3.2, the scheme-theoretic image of $\operatorname{Spec} A$ in $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ is also nowhere dense. By Lemma 4.5.6, we get a contradiction. □

Proof of Theorem 4.5.8. We will explicitly construct irreducible components of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i,\mathrm{red}}$ that exhaust all $\bar{\mathbb{F}}_p$-points of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i,\mathrm{red}}$, up to a nowhere dense subset.

By Lemma 4.5.9, an irreducible component C of $\coprod_i \mathcal{X}^{\mathrm{mns}}_{F,{}^LB,i,\mathrm{red}}$ determines an irreducible component Z_{H_M} of $\mathcal{X}_{F,{}^LH_M,\mathrm{red}}$.

Recall that $\mathcal{X}_{F,{}^LM,\mathrm{red}} = \mathcal{X}_{F,{}^L\mathrm{Res}_{K/F}\mathrm{G}_m,\mathrm{red}} \times \mathcal{X}_{F,{}^LH_M,\mathrm{red}} = \mathcal{X}_{K,\mathrm{G}_m,\mathrm{red}} \times \mathcal{X}_{F,{}^LH_M,\mathrm{red}}$.

Write ${}^LB_{H_M}$ for a Borel of ${}^LH_M$. By Lemma 4.5.6, Z_{H_M} is the scheme-theoretic image of an irreducible component C' of $\coprod \mathcal{X}^{\mathrm{mns}}_{F,{}^LB_{H_M},j,\mathrm{red}}$. Write Z for $\mathcal{X}_{K,\mathrm{G}_m,\mathrm{red}} \times C'$.

If H_M is not a torus, then H_M admits a niveau-1 maximal proper Levi subgroup of the form $\mathrm{Res}_{K/F}\mathrm{G}_m \times ?$ and there exists a projection ${}^LH_M \to {}^L\mathrm{Res}_{K/F}\mathrm{G}_m$. If H_M is a torus, then by our assumption, there exists a surjection $\mathrm{Res}_{K/F}\mathrm{G}_m \to H_M$, whose dual map is ${}^LH_M \to {}^L\mathrm{Res}_{K/F}\mathrm{G}_m$. Therefore, there exists a morphism $\mathcal{X}^{\mathrm{mns}}_{F,{}^LB_{H_M},j,\mathrm{red}} \to \mathcal{X}_{K,\mathrm{G}_m,\mathrm{red}}$. Denote by $\bar{\beta}$ the composition $C' \to \mathcal{X}^{\mathrm{mns}}_{F,{}^LB_{H_M},j,\mathrm{red}} \to \mathcal{X}_{K,\mathrm{G}_m,\mathrm{red}}$.

Similarly, there exists a morphism $\bar{\alpha} : Z \to \mathcal{X}_{K,\mathrm{G}_m,\mathrm{red}}$, and a morphism $x : \operatorname{Spec} \bar{\mathbb{F}}_p \to Z$ corresponds to an L-parameter of the form

$$(*) \qquad \begin{bmatrix} \bar{\alpha} & 0 & 0 & 0 \\ & \bar{\beta} & * & 0 \\ & & * & 0 \\ & & & \bar{\alpha}' \end{bmatrix} \rtimes *$$

Z can be decomposed into two locally closed substacks Z_i, $i = 0, 1$, defined so that

$$\dim H^2(\mathrm{Gal}_K, \bar{\alpha} \otimes \bar{\beta}^\vee) = i$$

over Z_i.

We first consider the relatively Steinberg irreducible components, whose scheme-theoretic image in $\mathcal{X}_{F,{}^LM,\mathrm{red}}$ will correspond to Z_1. First note that $Z_1 \cong [Z_{H_M}/\mathrm{G}_m]$ is irreducible by Lemma 4.5.2 (by local Tate duality, in the matrix presentation $(*)$, $\bar{\alpha} = \bar{\beta}(1)$ as Galois characters).

We can further decompose Z_1 into two locally closed substacks Z_{1i}, $i = 0, 1$, such that

$$\dim H^2(\mathrm{Gal}_K, \bar{\alpha} \otimes \bar{\alpha}'^\vee) = i$$

over Z_{1i}. Note that Z_{10} is either empty or dense in Z_1 by the semicontinuity theorem. If Z_{10} is dense in Z_1, then the preimage of Z_{11} in C is nowhere dense in C and we set $W_1 = Z_{10}$; it otherwise, set $W_1 = Z_{11}$. By Lemma 4.5.5, since W_1 is dense in Z_1, W_1 is irreducible. The morphism $W_1 \to \mathcal{X}_{F,{}^LM,\mathrm{red}}$ is U-basic, and the fiber product

$$W_1 \times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}} \mathcal{X}_{F,{}^LP}$$

parameterizes all L-parameters for G of the form

$$(**) \qquad \begin{bmatrix} \bar\alpha & * & * & * \\ & \bar\beta & * & * \\ & & * & * \\ & & & \bar\alpha' \end{bmatrix} \rtimes * = \begin{bmatrix} \bar\alpha & * & * \\ & \bar\tau & * \\ & & \bar\alpha' \end{bmatrix} \rtimes *$$

such that $H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\beta^\vee) \neq 0$, except for certain L-parameters parametrized by a nowhere dense subset of $\mathcal{X}_{F,{}^LG,\mathrm{red}}$. To show the relatively Steinberg components are in bijection with the components Z_{H_M}, it suffices to show $W_1 \times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}} \mathcal{X}_{F,{}^LP}$ is representable by an irreducible algebraic stack of finite type.

Before we proceed, we need the notion of *relative coarse moduli sheaf.* Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, define an sheaf $\mathcal{X}^{\mathrm{sh}}$ over $\mathrm{Sch}/\mathcal{Y}$ which sends $\mathrm{Spec}\, A \to \mathcal{Y}$ to the coarse moduli sheaf of $\mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}\, A$ ([L23B, 10.1.3]). We say $\mathcal{X}^{\mathrm{sh}}$ is the relative coarse moduli space of $\mathcal{X} \to \mathcal{Y}$ if $\mathcal{X}^{\mathrm{sh}} \to \mathcal{Y}$ is relatively representable by an algebraic space of finite type.

By [L23B, 10.1.5-10.1.7], we have

- the coarse moduli sheaf of
$$\mathcal{X}_{F,{}^LP/[U,U]} \times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}} W_1 \to W_1$$
is relatively representable by a vector bundle $Y_1 \to W_1$ of rank $\mathrm{rk}\, H^1(\mathrm{Gal}_F, U/[U,U])$,
- $\mathcal{X}_{F,{}^LP/[U,U]} \times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}} W_1 \cong [Y_1/H^0(\mathrm{Gal}_F, U/[U,U])]$
- there exists a closed substack $V_1 \subset Y_1$ such that the coarse moduli sheaf of
$$\mathcal{X}_{F,{}^LP} \times_{\mathcal{X}_{F,{}^LP/[U,U]}} Y_1 \to Y_1$$
is relatively representable by an affine bundle T_1 over V_1,
- $\mathcal{X}_{F,{}^LP} \times_{\mathcal{X}_{F,{}^LP/[U,U]}} Y_1 \cong [T_1/H^0(\mathrm{Gal}_F, U)]$.

In particular, $\mathcal{X}_{F,{}^LP} \times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}} W_1 \cong [[T_1 \times_{V_1} [V_1/H^0(\mathrm{Gal}_F, U/[U,U])]]/H^0(\mathrm{Gal}_F, [U,U])]$. By Lemma 4.5.2, it suffices to show T_1 is irreducible. Since W_1 is known to be irreducible, by Lemma 4.5.1, it remains to show all fibers of $V_1 \to W_1$ are irreducible of constant dimension (the dimension of the fibers is necessarily $\dim V_1 - \dim W_1$ because we have already known the dimension of $W_1 \times_{\mathcal{X}_{F,{}^LM,\mathrm{red}}} \mathcal{X}_{F,{}^LP}$ is equal to $\mathcal{X}_{F,{}^LG,\mathrm{red}}$ by the equidimensionality of the latter).

Before we finish off the proof of the relatively Steinberg case, we turn to the relatively non-Steinberg case since both cases can be dealt with uniformly. By the equidimensionality of Z, Z_1 is nowhere dense in Z and Z_0 is dense in Z. The scheme-theoretic image of a relatively non-Steinberg component in $\mathcal{X}_{F,{}^LM,\mathrm{red}}$ is an irreducible component of Z, or equivalently the closure of an irreducible component Z_{0x} of Z_0. Similarly, we can decompose Z_{0x} into two locally closed substacks Z_{0xi}, $i = 0, 1$, such that

$$\dim H^2(\mathrm{Gal}_K, \bar\alpha \otimes \bar\alpha'^\vee) = i$$

over Z_{0x}. By Definition 4.2.3(CS8), Z_{0x1} is nowhere dense in $\mathcal{X}_{F,{}^LM,\mathrm{red}}$ and thus nowhere dense in Z. Set $W_0 = Z_{0x0}$. Similarly define Y_0, V_0 and Y_0 by replacing W_1 by W_0 in the definition of Y_1, V_1 and Y_1.

Since $\dim H^2(\mathrm{Gal}_K, \bar{\alpha} \otimes \bar{\alpha}'^\vee) = 0$, $V_0 = Y_0$ is a vector bundle over W_0. The relatively non-Steinberg case is reduced to the claim that all fibers of $V_0 \to W_0$ are irreducible of constant dimension, which is clearly true since V_0 is a vector bundle over W_0.

Finally, we settle the relatively Steinberg case. By the proof of [L23B, Lemma 10.1.5], V_1 is the vanishing locus of a single quadratic form over the vector bundle Y_1. By Definition 4.2.3(CS4), the quadratic form is either pointwise trivial or pointwise non-trivial depending on if $\dim H^2(\mathrm{Gal}_K, \bar{\alpha} \otimes \bar{\alpha}'^\vee) = 0$ or 1. Thus the fibers of $V_1 \to W_1$ has constant dimension either $\dim H^1(\mathrm{Gal}_F, U/[U,U])$ or $\dim H^1(\mathrm{Gal}_F, U/[U,U]) - 1$. In the former case, $V_1 \to W_1$ is a vector bundle and we are done. In the latter case, the fibers of $V_1 \to W_1$ are the vanishing locus of a nontrivial quadratic form on a vector space of dimension $\dim H^1(\mathrm{Gal}_F, U/[U,U])$ and are irreducible if $\dim H^1(\mathrm{Gal}_F, U/[U,U]) \geq 3$ by Lemma 4.5.4. By the local Euler characteristic: $\dim H^1(\mathrm{Gal}_F, U/[U,U]) \geq \dim U/[U,U] + \dim H^2(\mathrm{Gal}_F, U/[U,U]) \geq 2 + 1 = 3$ since $\dim U/[U,U] \geq 2$, so we are done. □

5. The topological part of the geometric Breuil-Mézard conjecture: the unitary case

Write U_n for the quasi-split unitary group over F which splits over a quadratic extension K/F.

5.0.1. Lemma Assume $p \neq 2$. The identity map ${}^L U_n = \mathrm{GL}_n \rtimes \mathrm{Gal}(K/F)$ defines a classical structure in the sense of Definition 4.2.3.

Proof. (CS1, CS2, CS3, CS5) are clear: see [L23C, Theorem 2]. (CS4) is discussed in [L23C, Section 7]. The isomorphisms α, β in Definition 4.2.3(CS4) are given by the natural Galois involutions, see [L23C, Lemma 7.3].

(CS6, CS7, CS9): it is [L23C, Theorem 4]. (CS10): it is [L23C, Theorem 5].

(CS8) follows from (CS9) by induction on the rank of G. □

5.0.2. Theorem There exists a bijection between the irreducible components of $\mathcal{X}_{F, {}^L U_n, \mathrm{red}}$ and the irreducible components of $\mathcal{X}_{F, {}^L \mathrm{Res}_{K/F} \mathrm{G}_m \times U_{n-2}, \mathrm{red}} \amalg \mathcal{X}_{F, {}^L U_{n-2}, \mathrm{red}}$.

Proof. It follows immediately from Theorem 4.5.8. Note that $\dim U/[U,U] = 2(n-2) \geq 2$ if $n > 2$. □

In the case of even unitary groups, the irreducible components are in bijection with parabolic Serre weights.

5.0.3. Corollary Assume $F = \mathbb{Q}_p$. There exists a bijection between the irreducible components of $\mathcal{X}_{F, {}^L U_{2m}, \mathrm{red}}$ and the parahoric Serre weights for U_{2m}.

Proof. Recall that the superspecial parahoric $\underline{G}_m$ of U_{2m} is either U_{2m} or Sp_{2m} depending on whether U_{2m} is ramified or not. In either case, $\underline{G}_m$ has simply-connected derived subgroup and the weight lattice of $\underline{G}_m$ coincides with the coroot lattice of $\underline{G}_m$.

We refer to [L23, Section 6.1] for the notations for Serre weights.

We first consider the ramified case. Since $\underline{G}_m$ is semisimple, the set of (isomorphism classes of) parahoric Serre weights for U_{2m} are in bijection with p-restricted roots $X_1(\underline{T}_m)$ of $\underline{G}_m$. Here $\underline{T}_m$ is a maximal torus of $\underline{G}_m$. Write $\omega_1, \ldots, \omega_m$ for the fundamental weights of $\underline{G}_m$. Note that $\omega_1, \ldots, \omega_{m-1}$ are the fundamental weights of $\mathrm{Sp}_{2(m-1)}$, and thus $X_1(\underline{T}_m) = X_1(\underline{T}_{m-1}) \times \{0, 1, \ldots, p\}$. By induction on m, the irreducible components of $\mathcal{X}_{F, {}^L U_{2(m-1)}, \mathrm{red}}$ are in bijection with $X_1(\underline{T}_{m-1})$.

The irreducible components of $\mathcal{X}_{F,{}^L\mathrm{Res}_{K/F}\,\mathrm{G}_m,\mathrm{red}}$ correspond to inertial types $I_K \to \bar{\mathbb{F}}_p^\times$, which are powers of the fundamental character $\chi_{\mathrm{LT}}^i|_{I_K}$, $i = 0, \dots, p-1$. Here $\chi_{\mathrm{LT}} : \mathrm{Gal}_K \to \bar{\mathbb{F}}_p^\times$ is the Lubin-Tate character. We have thus identified the relatively non-Steinberg components of $\mathcal{X}_{F,{}^LU_{2m},\mathrm{red}}$ with $X_1(\underline{T}_{m-1}) \times \{0, 1, \dots, p-1\}$ and identified the relatively Steinberg components of $\mathcal{X}_{F,{}^LU_{2m},\mathrm{red}}$ with with $X_1(\underline{T}_{m-1}) \times \{p\}$ by Theorem 5.0.2.

Next, we consider the unramified case. Now $\underline{G}_m$ is not semisimple. The isomorphism classes of parahoric Serre weights for U_{2m} are in bijection with $X_1(\underline{T}_m)/(p-\pi)X^0(\underline{T}_m)$ where $\underline{T}_m$ is a Galois stable maximal torus of $\underline{G}_m$ and π is the Galois action on $\underline{T}_m$. Write $\omega_1, \dots, \omega_{2m-1}$ for the fundamental weights of $\underline{G}_m$. Note that $\omega_2, \dots, \omega_{2m-2}$ are the fundamental weights of $\mathrm{SU}_{2(m-1)}$. More concretely, the character lattice and the cocharacter lattice of $\underline{G}_m$ can both be identified with $\mathbb{Z}^{\oplus 2m}$. Write e_1, $\dots$, e_{2m} for the standard basis vectors of $\mathbb{Z}^{\oplus 2m}$ (so $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ whose only non-trivial entry is the i-th entry); the roots are $e_1 - e_2, e_2 - e_3, \dots, e_{2m-1} - e_{2m}$; the fundamental weights are e_1, $e_1 + e_2$, $\dots$, $e_1 + \cdots + e_{2m-1}$. We have $X_0(\underline{T}_m) = \mathbb{Z}(e_1 + e_2 + \cdots + e_{2m})$. Thus $X_1(\underline{T}_m)/(p-\pi)X_0(\underline{T}_m) \cong X_1(\underline{T}_{m-1})/(p-\pi)X_0(\underline{T}_{m-1}) \times \{0, \dots, p\} \times \{0, \dots, p\} \cong X_1(\underline{T}_{m-1})/(p-\pi)X_0(\underline{T}_{m=1}) \times \{0, \dots, p^2 - 1\}$. The irreducible components of $\mathcal{X}_{F,{}^L\mathrm{Res}_{K/F}\,\mathrm{G}_m,\mathrm{red}}$ correspond to inertial types $I_K \to \bar{\mathbb{F}}_p^\times$, which are powers of the fundamental character $\chi_{\mathrm{LT}}^i|_{I_K}$, $i = 0, \dots, p^2 - 1$. Here $\chi_{\mathrm{LT}} : \mathrm{Gal}_K \to \bar{\mathbb{F}}_p^\times$ is the Lubin-Tate character. We have thus identified the relatively non-Steinberg components of $\mathcal{X}_{F,{}^LU_{2m},\mathrm{red}}$ with $X_1(\underline{T}_{m-1}) \times \{0, 1, \dots, p^2 - 1\}$ and identify the relatively Steinberg components of $\mathcal{X}_{F,{}^LU_{2m},\mathrm{red}}$ with with $X_1(\underline{T}_{m-1}) \times \{p\}$ by Theorem 5.0.2. □

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