# EXTENSIONS OF CRYSTALLINE REPRESENTATIONS VALUED IN GENERAL REDUCTIVE GROUPS

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Abstract. We show a Galois representation valued in a parabolic subgroup of a reductive group is crystalline if it is crystalline modulo the unipotent radical, and has enough gaps in Hodge-Tate weights.

Contents



Let $K/\mathbb{Q}_p$ be a local field. Let $G_K$ be the absolute Galois group of $K$. Let $E$ be a finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$.

We prove the following:

**Theorem.** *(Corollary 2.4.3) Let $G$ be a connected reductive group. Let $P$ be a parabolic subgroup of $G_E$ with unipotent radical $U$ and Levi quotient $L$. Let $\rho : G_K \to L(E)$ be a crystalline Galois representation with enough gaps in Hodge-Tate weights with respect to $P$ (subsection 2.4.1). Then any parabolic lifting $\tilde{\rho} : G_K \to P(E)$ is crystalline.*

Here a parabolic lifting is a commutative diagram

$$\begin{array}{ccc} G_K & \xrightarrow{\tilde{\rho}} & P(E) \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & L(E) & \end{array}$$

We prove the theorem by reducing it to the $GL_N$-case via dynamic methods. The caveat is, the property of "having enough gaps in Hodge-Tate weights", is not preserved under dynamic methods. So we must pass to the category of weakly admissible filtered $\phi$-modules, and use dynamic methods indirectly.

## 1. Preliminaries

**1.1. $p$-adic Hodge theory** Let $E/\mathbb{Q}_p$ be a $p$-adic field with ring of integers $\mathcal{O}$. Let $G$ be a smooth connected group over $\mathcal{O}$. Let $\rho : G_K \to G(\mathcal{O})$ be a group homomorphism. We say $\rho$ is *crystalline* if for all algebraic representations $G \to GL_N$, the composition $G_K \to GL_N(\mathcal{O})$ is crystalline in the usual sense.

**1.1.1. Theorem** The representation $\rho$ is crystalline if and only if for some faithful embedding $G \to \mathrm{GL}_N$ and some finite extension $L/E$, the map $I_K \to G_K \xrightarrow{\rho} G(\mathcal{O}) \to G(\mathcal{O}_L) \to \mathrm{GL}_N(\mathcal{O}_L)$ is crystalline.

*Proof.* By [Le13, 5.3.2], we only need to check a single faithful embedding $G \hookrightarrow \mathrm{GL}_N$. By [BC08, 9.3.1], we only have to look at the inertia. □

**1.1.2. Lemma** Let $T$ be a smooth connected subgroup of $G$. Assume $\rho : G_K \to G(\mathcal{O})$ factors through $T(\mathcal{O})$. Then $\rho$ is crystalline as a $G$-valued representation if and only if it is crystalline as a $T$-valued representation.

*Proof.* Choose an embedding $G \hookrightarrow \mathrm{GL}_N$. The lemma follows by applying the theorem above twice. □

**1.2. Exact $\otimes$-filtrations** We review notions that are necessary for our general construction.

Let $\mathscr{C}$ be an ind-tannakian category ([SN72, III 1.1.1]) over a ring $A$. Let $\mathrm{Vect}_A$ be the category of projective $A$-modules. Let $\omega : \mathscr{C} \to \mathrm{Vect}_A$ be an exact tensor functor. For $X \in \mathrm{Vect}_A$, a filtration of $X$ indexed by $\mathbb{Z}$ is an tuple $(\mathrm{Fil}^n X)_{n\in\mathbb{Z}}$ where $\mathrm{Fil}^n X \in \mathrm{Vect}_A$, $\mathrm{Fil}^n X \supset \mathrm{Fil}^{n+1} X$, $\cap \mathrm{Fil}^n X = 0$, and $\cup \mathrm{Fil}^n X = X$. Let $\mathrm{Fil}\,\mathrm{Vect}_A$ be the category of filtered (indexed by $\mathbb{Z}$) projective $A$-modules.

An *exact $\otimes$-filtration* $F$ on $\omega$ is a factorization

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad\omega\quad} & \mathrm{Vect}_A \\ & {\scriptstyle\tilde{\omega}}\searrow \quad \nearrow{\scriptstyle \text{forget}} & \\ & \mathrm{Fil}\,\mathrm{Vect}_A & \end{array}$$

such that the following are satisfied:

(FE 1) For $X \in \mathscr{C}$, $\mathrm{Fil}^n \tilde{\omega}(X)$ is a direct summand of $\omega(X)$;
(FE 2) The associated graded functor $\mathrm{gr}_F(\tilde{\omega})$ is exact.
(FE 3) For all $n \in \mathbb{Z}$, $X, Y \in \mathscr{C}$,

$$\mathrm{Fil}^n \tilde{\omega}(X \otimes Y) = \sum_{i+j=n} \mathrm{Fil}^i \tilde{\omega}(X) \otimes \mathrm{Fil}^j \tilde{\omega}(Y).$$

Let $\omega$, $\omega'$ be exact tensor functors with exact $\otimes$-filtrations $F$, $F'$, respectively. Denote by

$$\text{Isom-fil}^{\otimes}((\omega, F), (\omega', F'))$$

the functor of tensor isomorphisms inducing an isomorphism of filtrations. Set

$$\mathrm{Aut}^{\otimes}(F) := \text{Isom-fil}^{\otimes}((\omega, F), (\omega, F))$$

for simplicity of notation. Denote by $\mathrm{Aut}^{\otimes !}(F)$ the subfunctor of $\mathrm{Aut}^{\otimes}(F)$ which induces the identity of the associated grading.

**1.3. Splitting of exact $\otimes$-filtrations** Let $\mathrm{gr}\,\mathrm{Vect}_A$ be the category of graded vector spaces. An exact tensor functor $\omega : \mathscr{C} \to \mathrm{gr}\,\mathrm{Vect}_A$ induces a canonical exact $\otimes$-filtration $F_{\mathrm{can}}$, which is defined as $\mathrm{Fil}^n(\omega(X)) := \sum_{n' \geq n} \mathrm{gr}_{n'}\, \omega(X)$ for all $X \in \mathscr{C}$.

An exact $\otimes$-filtration is said to be *splittable* if it is isomorphic to the canonical exact $\otimes$-filtration associated to a graded exact tensor functor.

## 2. Extension of weakly admissible filtered $\phi$-$G$-torsors

### 2.1. Filtered $\phi$-$G$-torsors and crystalline representations

**2.1.1. Definition** Let $K/\mathbb{Q}_p$ be a finite extension. Let $K_0 = W(k)[1/p]$ where $k$ is the residue field of $K$. Let $E$ be a sufficiently large coefficient field (admitting an embedding of the normal closure of $K$). A *filtered $\phi$-module with coefficients in $E$* is a triple $(D, \phi_D, \theta_D)$ where

- $D$ is a finite free module over $K_0 \otimes_{\mathbb{Q}_p} E$;
- $\phi_D : (\phi \otimes 1)^* D \to D$ is an isomorphism of $K_0 \otimes_{\mathbb{Q}_p} E$-modules;
- $\theta_D$ is a filtration on $D_K := D \otimes_{K_0} K$ such that $\theta_D^j D_K = 0$ if $j \gg 0$, and $\theta_D^j D_K = D_K$ if $j \ll 0$.

Here $\phi \otimes 1 : K_0 \otimes E \to K_0 \otimes E$ sends $x \otimes y$ to $\phi(x) \otimes y$.

**2.1.2. Definition** A *filtered $\phi$-$G$-torsor with coefficients in $E$* is a triple $(T, \phi_T, \theta_T)$ such that

- $T$ is a $G$-torsor over $\mathrm{Spec}\, K_0 \otimes E$;
- $\phi_T : (\phi \otimes 1)^* T \to T$ is a $G$-equivariant isomorphism over $\mathrm{Spec}\, K_0 \otimes_{\mathbb{Q}_p} E$;
- $\theta_T$ is an exact $\otimes$-filtration on $T_K := T \underset{\mathrm{Spec}\, K_0 \otimes_{\mathbb{Q}_p} E}{\times} \mathrm{Spec}\, K \otimes_{\mathbb{Q}_p} E$.

More precisely, $\theta_T$ is an exact $\otimes$-filtration on the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{K \otimes E}$ defined by $V \mapsto T_K \times^G V$. By Tannakian theory, a $G$-torsor always comes from a rigid exact $\otimes$-functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{K \otimes E}$, so we don't distinguish them.

**2.1.3. Remark** (1) We can define the notion of filtered $\phi$-$G$-torsor with coefficient in $E$ for any smooth $E$-group scheme $G$.

(2) By Tannakian theory and the functoriality of the twisted product $- \times^G -$, a filtered $\phi$-$G$-torsor $(T, \phi_T, \theta_T)$ is nothing but a rigid exact tensor functor from $\mathrm{Rep}_G(E)$ to the category of filtered $\phi$-modules with coefficients in $E$.

For ease of notation, we write $- \otimes -$ for $- \otimes_{\mathbb{Q}_p} -$.

**2.1.4. Pushforward** Let $f : G \to H$ be a group scheme morphism.

Let $(T, \phi_T, \theta_T)$ be a filtered $\phi$-$G$-torsor. Define $T' := T \times^G H := T \times H / \{(t, h) \sim (g^{-1} \cdot t, f(g) \cdot h), g \in G, t \in T, h \in H\}$. Then $T'$ is an $H$-torsor with $H$-action defined by $h \cdot (t, h') = (t, hh')$. Since $(\phi \otimes 1)^* T \times^G H \cong (\phi \otimes 1)^* (T \times^G H)$ canonically, we can define $\phi_{T'} := \phi \times^G H : (\phi \otimes 1)^* T' \to T'$. Recall that $\theta_T$ is a functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{K \otimes E}$. Define $\theta_{T'} := f_*(\theta_T)$ to be the composite $\mathrm{Rep}(H) \xrightarrow{f^*} \mathrm{Rep}(G) \to \mathrm{Vect}_{K \otimes E}$. The triple $(T', \phi_{T'}, \theta_{T'})$ is a filtered $\phi$-$H$-torsor. We write $f_*(T, \phi_T, \theta_T)$ for $(T', \phi_{T'}, \theta_{T'})$.

**2.1.5. Framing** Let $(T, \phi_T, \theta_T)$ be a filtered $\phi$-$G$-torsor. Suppose the underlying $G$-torsor $T$ is a trivial $G$-torsor, there exists a canonical embedding

$$\iota : T(K_0 \otimes E) \hookrightarrow T(K_0 \otimes E) \times_{\{\mathrm{pt}\},\phi\otimes 1} \{\mathrm{pt}\} = (\phi \otimes 1)^* T(K_0 \otimes E), \quad \{\mathrm{pt}\} = (\operatorname{Spec} K_0 \otimes E)(K_0 \otimes E)$$

A *framing* of $T$ is an element $\xi \in T(K_0 \otimes E)$. Since $T$ is a $G$-torsor, there exists a unique element $X_\xi \in G(K_0 \otimes E)$ such that $\phi_T(\iota(\xi)) = X_\xi \cdot \xi$.

Let $g \in G(K_0 \otimes E)$. Now we change the framing from $\xi$ to $g \cdot \xi$. We have $\phi_T(\iota(g \cdot \xi)) = \phi(g)\phi_T(\iota(\xi)) = \phi(g)X_\xi \cdot \xi = \phi(g)X_\xi g^{-1} g \cdot \xi$. Therefore

$$X_{g\cdot\xi} = \phi(g)X_\xi g^{-1}.$$

Let $f : G \to H$ be a group scheme homomorphism. Let $\xi \in T(K_0 \otimes E)$ be a framing. Then $f_*(\xi) \in (T \times^G H)(K_0 \otimes E)$ is a framing of $f_*(T, \phi_T, \theta_T)$. It is easy to see that

$$X_{f_*\xi} = f(X_\xi).$$

**2.1.6. Weak admissibility** For simplicity, we define the weak admisibility of a filtered $\phi$-$G$-torsor via Tannakian theory. A filtered $\phi$-$G$-torsor $T$ is *weakly admissible* if for any algebraic representation $G \to GL(V)$, the twisted product $T \times^G V$ is a weakly admissible filtered $\phi$-module.

**2.1.7. Crystalline representations** Since the covariant Fontaine's functors $V_{\mathrm{cris}}$ and $D_{\mathrm{cris}}$ are rigid exact tensor functors (see the paragraph before [C11, 9.1.9]), the category of weakly admissible filtered $\phi$-$G$-torsors is equivalent to the category of crystalline representations valued in $G$. We also denote by $V_{\mathrm{cris}}$ and $D_{\mathrm{cris}}$ the equivalences of categories in the $G$-valued case.

### 2.2. Parabolic liftings

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$ and Levi factor $L$. Let $\pi_L : P \to L$ be the quotient map. Let $(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}})$ be a fixed filtered $\phi$-$L$-torsor with coefficients in $E$.

Define

$$\begin{aligned} \mathrm{Lift}(\bar{T}) &= \mathrm{Lift}(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}}) \\ &= \{(T, \phi_T, \theta_T) : \textit{filtered } \phi\textit{-}P\textit{-torsors valued in } E \textit{ such that } (\pi_L)_* T = \bar{T}\}/\sim \end{aligned}$$

where the equivalence relation $\sim$ is defined to be isomorphisms of filtered $\phi$-$P$-torsors respecting $(\pi_L)_*$.

**2.2.1.** Throughout this section, we assume $\bar{T}$ admits a framing $\bar{\xi} \in \bar{T}(K_0 \otimes E)$. By fixing $\bar{\xi}$, we also fixed a framing of $\iota_* \bar{T}$ for various sections $\iota : L \hookrightarrow P$.

In particular, for two different sections $\iota_1$, $\iota_2 : L \to P$, the two $P$-torsors $\bar{T} \times^{L,\iota_1} P$ and $\bar{T} \times^{L,\iota_2} P$ are identified without further mention.

Moreover, since the base scheme is a disjoint union of spectra of perfect fields, any element of $\mathrm{Lift}(\bar{T})$ admits a framing ([Se02, Proposition 6, III.2.1]).

**2.2.2. Lemma** Let $T \in \mathrm{Lift}(\bar{T})$. Then there exists a section $\iota : L_{K\otimes E} \hookrightarrow P_{K\otimes E}$ such that

$$\theta_T = \iota_*(\theta_{\bar{T}})$$

($\iota$ is a group scheme morphism and $\pi_L \circ \iota = \mathrm{id}$.)

*Proof.* Since $G$ is smooth and the coefficient ring is of characteristic 0, the exact $\otimes$-filtration $\theta_T$ is Zariski-locally splitable on $\mathrm{Spec}\, K \otimes E$ ([SN72, IV.2.4]). Since $K \otimes E$ is a direct sum of fields, $\theta_T$ is splittable. So $\theta_T$ is the canonical filtration associated to a graded tensor functor, or equivalently a cocharacter $\omega : (\mathbb{G}_m)_{K\otimes E} \to P_{K\otimes E}$ ([SN72, IV.1.3]). We choose an arbitrary embedding $L_{K\otimes E} \subset P_{K\otimes E}$. The image of $\omega$ is contained in a maximal torus, and hence contained in a conjugate of $L_{K\otimes E}$ (say $L_{K\otimes E} = \coprod_{i:K\hookrightarrow E} L_{E_i}$, the image of $\omega \otimes E_i$ is contained in a conjugate of $L_{E_i}$).

Choose a section $\iota : L_{K\otimes E} \to P_{K\otimes E}$ such that $\omega(\mathbb{G}_m) \subset \iota(L_{K\otimes E})$. We have $\iota_*(\pi_L)_*(\omega) = \omega$. Now it is clear that $\theta_{\bar{T}}$ is the canonical exact $\otimes$-filtration associated to $(\pi_L)_*(\omega)$, and $\theta_T = \iota_*(\theta_{\bar{T}})$ □.

**2.2.3. The adjoint filtered $\phi$-module** Recall that the upper central series of $U$ defines a filtration

$$\{1\} = U_s \subset U_{s-1} \subset \cdots \subset U_0 = U$$

such that each of $\mathrm{gr}_i\, U := U_i/U_{i+1}$ is abelian. We have

$$\mathrm{Lie}\, U = \bigoplus_{i=1}^{s} \mathrm{gr}_i\, U = \mathrm{gr}_\bullet\, U$$

Since $P = L \ltimes U$, a section $L \to P$ induces an (adjoint) action $L \curvearrowright U$. Let $\mathrm{ad} : L \to \mathrm{Aut}(U)$ be the induced group scheme homomorphism. Note that the abelianization $\mathrm{gr}^\bullet(\mathrm{ad}) : L \to \mathrm{Aut}(\mathrm{Lie}\, U)$ does not depend on the choice of $L \to P$.

Define

$$\mathrm{gr}^\bullet(\mathrm{ad})\bar{T} := \mathrm{gr}^\bullet(\mathrm{ad})_*(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}})$$

**2.2.4. Lemma** If $\bar{T}$ is weakly admissible, then so is $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$.

*Proof.* By Tannakian theory (more precisely by the fact that the Tannakian category is generated by $\rho \otimes \rho^*$ where $\rho$ is any faithful representation of the Tannakian group), a Galois representation $G_K \to \mathrm{Aut}(\mathrm{Lie}\, U)(E)$ is crystalline if and only if for some faithful algebraic representation $\mathrm{Aut}(\mathrm{Lie}\, U) \to \mathrm{GL}(V)$ the representation $G_K \to \mathrm{GL}(V(E))$ is crystalline. Since the composition $L \to \mathrm{Aut}\,\mathrm{Lie}(U) \to \mathrm{GL}(V)$ is an algebraic representation, $G_K \to \mathrm{GL}(V(E))$ is crystalline if $G_K \to L(E)$ is crystalline. The first claim is proved by passing to the category of crystalline representations via $V_{\mathrm{cris}}$. □

## 2.3. $G$-ordinarity

**2.3.1. Newton polygon of isocrystals** Let $\breve{K}$ be the $p$-adic completion of the maximal unramified extension of $K_0$. By the Diedonné-Manin classification, the category of isocrystals over $\breve{K}$ is a semisimple category. The simple objects can be classified by rational numbers $s/r$, where $r$ is a positive integer and $s$ is an integer coprime to $r$. Denote by $D_{r,s}$ the simple object labeled by the rational number $s/r$. $D_{r,s}$ has dimension $r$, and we call $s/r$ the *slope* of $D_{r,s}$.

Let $(D, \phi)$ be an isocrystal over $K_0$. Then $\breve{D} = \breve{K} \otimes_{K_0} D$ is a direct sum of simple objects $D_{r_i,s_i}$. We call the numbers $s_i/r_i$ that appear in the direct sum decomposition the slopes of $D$. Say $D$ has slopes $\{\alpha_0 < \cdots < \alpha_n\}$ with multiplicities $\{\mu_0, \cdots, \mu_n\}$. The *Newton polygon* of $D$ is the convex polygon with leftmost endpoint $(0, 0)$, and having $\mu_i$ consecutive segments of horizontal distance 1 and slope $\alpha_i$.

**Lemma** If all slopes of $D$ are positive numbers, then for any lattice $\mathcal{L} \subset D$, we have

$$\lim_{n\to\infty} \phi^n \mathcal{L} = \{0\}$$

(in the sense that the diameter of the bounded sets $\phi^n\mathcal{L}$ converges to 0.) Note that $\mathcal{L}$ is not assumed to be $\phi$-stable.

*Proof.* Let $N$ be the product of the denominator of the slopes of $D$. By the Diedonné-Manin classification, there is a basis $\{x_1, .., x_t\}$ of $\breve{K} \otimes_{K_0} D$ such that $\phi^N x_i = p^{S_i} x_i$ for positive integers $S_i$, $1 \leq i \leq t$. The $p$-adic topology on $D$ can be defined by

$$|\lambda_1 x_1 + \cdots + \lambda_t x_t| = \max_{1\leq i\leq t}(|\lambda_i|)$$

where $\lambda_i \in \breve{K}$, $1 \leq i \leq t$. For any $x \in D$, $|\phi^N x| < \frac{1}{p}|x|$. Therefore for any lattice $\mathcal{L}$, we have $\lim_{n\to\infty} \phi^{nN}\mathcal{L} = \{0\}$. Replacing $\mathcal{L}$ by $\phi^k\mathcal{L}$, $1 \leq k < N$, we have $\lim_{n\to\infty} \phi^{k+nN}\mathcal{L} = \{0\}$. Combining these, we have $\lim_{n\to\infty} \phi^n\mathcal{L} = \{0\}$. □

**Corollary** If the slopes of $D$ are either all positive numbers or all negative numbers, the map $1 - \phi : D \to D$ is invertible.

*Proof.* If the slopes of $D$ are all positive numbers, then $1 + \phi + \phi^2 + \cdots$ converges and is an inverse of $1 - \phi$.

If the slopes of $D$ are all negative numbers, then the slopes of the dual isocrystal $D^\vee$ are all positive numbers. By choosing a basis of $D$, the matrix of $\phi^\vee$ is the transpose inverse of that of $\phi$, and $(1 - \phi^{-t}) = -\phi^{-t}(1 - \phi^t)^{-1} = -\phi^{-t}(1 + \phi^t + \phi^{2t} + \cdots)$ is invertible. □

**2.3.2. Definition** Let $(T, \phi_T, \theta_T)$ be a filtered $\phi$-$P$-torsor. $\bar{T} = (\pi_L)_*T$ is a filtered $\phi$-$L$-torsor. Note that since $\mathrm{Aut}(\mathrm{Lie}\, U)$ is a general linear group, $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})$ is an filtered isocrystal with coefficients.

We say $\theta_T$, $T$, or $\bar{T}$ is *G-ordinary* if the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on the vector space $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})$ satisfies $\theta^0_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}(\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}) = 0$. In other words, the Hodge polygon of $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}$ lies below the $x$-axis except for the left endpoint.

**Lemma** A weakly admissible filtered $\phi$-$G$ torsor $T$ is $G$-ordinary if and only if all Hodge-Tate weights of the crystalline representation $V_{\mathrm{cris}}(\mathrm{gr}^\bullet(\mathrm{ad})\bar{T})$ are negative integers, where $V_{\mathrm{cris}}$ is the covariant Fontaine functor.

*Proof.* It is a standard $p$-adic Hodge theory calculation. See for example [C11, section 8.3]. □

**2.3.3. Dynamic methods** We need some results from [Crd11, Section 4.1]. Let $X$ be a scheme over a base scheme $S$, and fix a $\mathbb{G}_m$-action $m : \mathbb{G}_m \times X \to X$ on $X$. For each $x \in S(S)$, we say

$$\lim_{t\to 0} m(t,x) \quad \text{exists,}$$

if the morphism $\mathbb{G}_m \to X$, $t \mapsto m(t,x)$ extends a a morphism $\mathbb{A}^1 \to X$.

Let $\lambda$ be a cocharacter of a reductive group $G$. Define the following functor on the category of $K \otimes E$-algebras $P_G(\lambda)(k) = \{g \in G(k) |\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists.$\}$ where $k$ is a general $K \otimes E$-algebra.

$P_G(\lambda)$ is a smooth subgroup of $G$, and all parabolic subgroups of $G$ are of the form $P_G(\lambda)$ for some $\lambda$.

Define $U_G(\lambda)(k) = \{g \in G(k) |\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\}$. Then $U_G(\lambda) \subset P_G(\lambda)$ is the unipotent radical.

Denote by $L_G(\lambda)$ the quotient $P_G(\lambda)/U_G(\lambda)$.

Let $f : G \to H$ be a group scheme morphism. We have induced group scheme morphisms $P_G(f) : P_G(\lambda) \to P_H(f_*\lambda)$ and $U_G(f) : U_G(\lambda) \to U_H(f_*\lambda)$ ([Crd11, Theorem 4.1.7]).

A cocharacter $\lambda$ of $G$ induces a filtration $F(\lambda)$ on the trivial $G$-torsor ([SN72, IV 2.1.5]).

**Theorem** Consider the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$. We have

$$\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(F(\lambda)) = F(\lambda)^0(\mathrm{Lie}(G))$$

and

$$\mathrm{Lie}\,\mathrm{Aut}^{\otimes !}(F(\lambda)) = F(\lambda)^1(\mathrm{Lie}(G)).$$

As a consequence, we have $P_G(\lambda) = \mathrm{Aut}^{\otimes}(F(\lambda))$ and $U_G(\lambda) = \mathrm{Aut}^{\otimes !}(F(\lambda))$.

*Proof.* The first paragraph is a special case of [SN72, IV 2.1.4.1] where $\alpha = 0, 1$. The second paragraph follows from [Crd11, Theorem 4.1.7(4)]. □

**2.3.4.** Suppose $P_{K\otimes E} = P_G(\lambda)$ for some cocharacter $\lambda$ of $G$. The cocharacter $\lambda$ induces a filtration $F(\lambda)$ on $G_{K\otimes E}$.

Let $(T, \phi_T, \theta_T)$ be a $G$-ordinary filtered $\phi$-$P$-torsor whose underlying $G$-torsor is a trivial $G$-torsor. By Lemma 2.2.2, there exists an embedding $\iota : L_{K\otimes E} \to P_{K\otimes E}$ such that $\theta_T = \iota_*(\theta_{\bar{T}})$. We'll explicitly construct $\iota$ when $T$ is $G$-ordinary and show that such an embedding is unique. Write $i$ for the embedding $P_{K\otimes E} \to G_{K\otimes E}$.

**Proposition** There exists a unique embedding $\iota : L_{K\otimes E} \to P_{K\otimes E}$ such that $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset \iota(L_{K\otimes E})$.

*Proof.* The intersection of two parabolics of a reductive group always contains a maximal torus ([M18, 19.33]). Let $S \subset \mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E}$ be a maximal torus. Let $S_0 \subset S$ be the maximal subtorus such that the centralizer $Z(S_0)$ is (an embedding of) the Levi factor $L_{K\otimes E}$ of $P_{K\otimes E}$.

Let $U_{K\otimes E}$ be the unipotent radical of $P_{K\otimes E}$. We have the Levi decomposition

$$\mathrm{Lie}\, P_{K\otimes E} = \mathrm{Lie}\, Z(S_0) \oplus \mathrm{Lie}(U_{K\otimes E}) = \mathrm{Lie}\, Z(S_0) \oplus \bigoplus_{\alpha\in\Phi^+(S_0,G)} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha$ is the $\alpha$-weight space and $\Phi^+(S_0, G)$ is the set of weights occurring in $\mathrm{Lie}(U_{K\otimes E})$.

By Theorem 2.3.3, $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) = (i_*\theta_T)^0(\mathrm{Lie}\, G)$. Since $T$ is $G$-ordinary, we have $\theta^0_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}(\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})) = 0$. It is clear that the filtration $i_*\theta_T$ on $\mathrm{Lie}\, G$ and the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T}) = \mathrm{Lie}(U_{K\otimes E})$ are compatible. So $i_*\theta^0_T(\mathrm{Lie}\, G) \cap \mathrm{Lie}(U_{K\otimes E}) = 0$.

Consider the $S_0$-weight decomposition of $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap \mathrm{Lie}\, P_{K\otimes E}$. By the previous paragraph, there is no positive $S_0$-weights, and therefore $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap \mathrm{Lie}\, P_{K\otimes E} \subset \mathrm{Lie}\, Z(S_0)$. So we've shown $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset Z(S_0)$.

It remains to show the uniqueness of $\iota$. Let $g \in P(K\otimes E)$ and suppose $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset gZ(S_0)g^{-1}$. The Proposition follows from the fact that $S_0 \subset gZ(S_0)g^{-1}$ implies $gZ(S_0)g^{-1} = Z(S_0)$. □

**Lemma** If $\theta_T = \iota_*\theta_{\bar{T}}$, then $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset \iota(L_{K\otimes E})$.

*Proof.* Choose a splitting $\omega$ of $\theta_{\bar{T}}$. Choose a maximal torus $S$ of the centralizer of $\omega$. By Theorem 2.3.3 and Theorem [Crd11, 4.1.7(4)], we have

$$\mathrm{Aut}^{\otimes}(i_*\theta_T) = \mathrm{Lie}\, P_G(\omega) = \bigoplus_{\langle\alpha,\omega\rangle\geq 0} \mathfrak{g}_\alpha$$

where $\alpha$ ranges from all $S$-roots of $G$, and $\mathfrak{g}_\alpha$ is the $S$-weight space of weight $\alpha$. Meanwhile, $P_{K\otimes E} = P_G(\lambda)$ for some cocharacter $\lambda : \mathbb{G}_m \to S$. Since $T$ is $G$-ordinary, $\theta^0_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}(\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})) = 0$, which implies for any root $\alpha \in \Phi(S, G)$ such that $\langle\alpha, \lambda\rangle > 0$ we have $\langle\alpha, \omega\rangle < 0$. This lemma now follows from the $S$-weight decomposition of the Lie algebra of $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E}$. □

**2.3.5. Lemma** If $T$ is $G$-ordinary, the map $1 - \phi_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}$ is invertible.

*Proof.* By Corallary 2.3.1, it suffices to show all slopes of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ are negative numbers.

The $G$-ordinarity condition guarantees the Hodge polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ lies below the $x$-axis (except for the left endpoint which is the origin). Weak admissibility of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ implies the Newton polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ and the Hodge polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ have the same right endpoint, and the Newton polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ lies on or above the Hodge polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$. In particular, the largest slope of the Newton polygon is smaller or equal to the largest slope of the Hodge polygon. In other words, all slopes of the Newton polygon are negative numbers. □

**2.3.6. Lemma** Let $T$ be a filtered $\phi$-$P$-torsor. Assume $T$ is weakly admissible and $G$-ordinary.

A section $\iota : L_{K_0\otimes E} \hookrightarrow P_{K_0\otimes E}$ induces a projection $P_{K_0\otimes E} \xrightarrow{\pi_U} U_{K_0\otimes E}$ and a decomposition $P_{K_0\otimes E} = L_{K_0\otimes E}U_{K_0\otimes E}$.

There exists a unique section $\iota$ such that $\phi_T = \iota_*(\phi_{\bar{T}})$. (See 2.2.1.)

*Proof.* For ease of notation, write $L$, $P$, $U$ for $L_{K_0\otimes E}$, $P_{K_0\otimes E}$, and $U_{K_0\otimes E}$, respectively. Fix a framing $\xi$ of $T$. Write $\bar{T}$ for $(\pi_L)_*T$.

We have fixed a framing $\bar{\xi} \in L(K_0 \otimes E)$ of $\bar{T}$. Choose a section $\iota_0 : L \hookrightarrow P$, which induces a projection $P \xrightarrow{\pi_{U,0}} U$. There exists a unique isomorphism of $P$-torsors $(\iota_0)_*\bar{T} \cong T$ under which $(\iota_0)_*\bar{\xi}$ is identified with $\xi$. We identify $(\iota_0)_*\bar{T}$ and $T$ via this isomorphism. By remark 2.1.4, $\phi_T = (\iota_0)_*(\phi_{\bar{T}})$ if and only if $X_\xi = \iota_0(X_{\bar{\xi}})$. Or equivalently, $\pi_{U,0}(X_\xi) = 1$.

Set $M_0 := \pi_{U,0}(X_\xi)$, and $A_0 := \iota_0(\pi_L(X_\xi))$. Then $X_\xi = A_0M_0$ (we identify $U$ as a normal subgroup of $P$). Let $\iota : L \hookrightarrow P$ be another section with induced projection $P \xrightarrow{\pi_U} U$. Set $A := \iota(\pi_L(X_\xi))$ and $M := \pi_U(X_\xi)$. Note that there exists $N \in U(K_0 \otimes E)$ such that $A = NA_0N^{-1}$ ([SN72, IV 2.2.5.3]). Since

$$AM = A_0M_0 = X_\xi$$

we have $NA_0N^{-1}M = A_0(A_0^{-1}NA_0N^{-1}M) = A_0M_0$, and thus

$$M_0 = \mathrm{Ad}_{A_0^{-1}}(N)N^{-1}M.$$

For ease of notation, we write $\phi_{\mathrm{Ad}} := \mathrm{Ad}_{A_0^{-1}}$, and $M_0 = \phi_{\mathrm{Ad}}(N)N^{-1}M$.

For an integer $1 \le i \le s$, write $\mathrm{gr}^i$ for the projection $U_{i-1} \to U_{i-1}/U_i$. We use additive notation when working with abelian groups.

We have

$$\mathrm{gr}_1 M = (1 - \phi_{\mathrm{Ad}})\,\mathrm{gr}_1(N) + \mathrm{gr}_1 M_0.$$

Note that $(1 - \phi_{\mathrm{Ad}})\,\mathrm{gr}_1(N) = (1 - \phi^{-1}_{\mathrm{gr}_\bullet \mathrm{ad}\,\bar{T}})\,\mathrm{gr}_1(N)$. By Lemma 2.3.5, $(1 - \phi_{\mathrm{Ad}}) : \mathrm{Lie}\, U \to \mathrm{Lie}\, U$ is invertible. We choose $N \in U$ such $\mathrm{gr}_1(N) = (\phi_{\mathrm{Ad}} - 1)^{-1}M_0$ (choosing $\iota$ is equivalent to choosing $N$). Hence we can arrange it so that $\mathrm{gr}_1 M = 0$.

Now we assume $\mathrm{gr}_1 M_0 = 0$, that is, $M_0 \in U_1$. We choose $\iota$ such that $N \in U_1$. We have

$$\mathrm{gr}_2 M = (1 - \phi_{\mathrm{Ad}})\mathrm{gr}_2(N) + \mathrm{gr}_2 M_0.$$

We can kill $\mathrm{gr}_2(M)$ in a similar manner. We repeat this process, and will ultimately kill $M$.

The uniqueness of $\iota$ is a byproduct of the proof of the existence part. In each step of the above algorithm, the choice is unique. □

Denote by $\mathrm{Scin}(P)$ the set of sections $L \hookrightarrow P$. Note that $\mathrm{Scin}(P)$ is a $U$-torsor ([SN72, IV 2.2.5.3]).

**2.3.7.** Let $(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}})$ be a filtered $\phi$-$L$-torsor which is weakly admissible and $G$-ordinary (with respect to the parabolic $P$).

The following map

$$\delta : \mathrm{Scin}(P_{K_0\otimes E}) \times \mathrm{Scin}(P_{K\otimes E}) \to \mathrm{Lift}(\bar{T})$$
$$(\iota_\phi, \iota_\theta) \mapsto (T, (\iota_\phi)_*\phi_{\bar{T}}, (\iota_\theta)_*\theta_{\bar{T}})$$

is a surjection by Lemma 2.3.6 and Lemma 2.2.2.

Note that $U(K_0 \otimes E)$ acts diagonally on $\mathrm{Scin}(P_{K_0\otimes E}) \times \mathrm{Scin}(P_{K\otimes E})$.

**Theorem** We have bijections

$$\begin{aligned} \mathrm{Lift}(\bar{T}) &\cong \{\text{Orbits of } \mathrm{Scin}(P_{K_0\otimes E}) \times \mathrm{Scin}(P_{K\otimes E}) \text{ under } U(K_0\otimes E) \text{ action}\} \\ &\cong \mathrm{Scin}(P_{K\otimes E}) \end{aligned}$$

*Proof.* We only need to show two liftings of $\bar{T}$ are equivalent if and only if they lie in the same $U_{K_0\otimes E}$-orbit.

Let $\delta(\iota_\phi^1, \iota_\theta^1)$ and $\delta(\iota_\phi^2, \iota_\theta^2)$ be two liftings. Let $h : \delta(\iota_\phi^1, \iota_\theta^1) \cong \delta(\iota_\phi^2, \iota_\theta^2)$ be an isomorphism of filtered $\phi$-$G$-torsors whose pushforward along $\pi_L$ is the identity map. Identify the underlying trivial $P_{K_0\otimes E}$-torsor with $P_{K_0\otimes E}$. Then $h$ is just conjugation by an element $u$ of the unipotent radical $U(K_0\otimes E)$. In particular,

$$(\iota_\phi^1)_*\phi_{\bar{T}} = (u\iota_\phi^2 u^{-1})_*\phi_{\bar{T}}, \quad (\iota_\theta^1)_*\theta_{\bar{T}} = (u\iota_\theta^2 u^{-1})_*\theta_{\bar{T}}.$$

By Proposition 2.3.4 and Lemma 2.3.4, two different sections $\iota_\theta : L_{K\otimes E} \hookrightarrow P_{K\otimes E}$ gives two different filtrations $(\iota_\theta)_*\theta_{\bar{T}}$. Therefore we have $\iota_\theta^1 = u\iota_\theta^2 u^{-1}$. Similarly, by Lemma 2.3.6, we have $\iota_\phi^1 = u\iota_\phi^2 u^{-1}$. So $(\iota_\phi^1, \iota_\theta^1)$ and $(\iota_\phi^2, \iota_\theta^2)$ are in the same $U$-orbit, as desired. □

The theorem above is reminiscent of the double complex computing the cohomology of filtered $\phi$-modules.

**2.3.8.** Recall

$$\{1\} = U_s \subset U_{s-1} \subset \cdots \subset U_0 = U$$

is the upper central series of $U$.

**Corollary** Let $T_i$ be a filtered $\phi$-$P/U_i$-torsor for some $1 \le i \le s$, which can be lifted to a filtered $\phi$-$P$-torsor. Assume $\bar{T} := T_i \mod U/U_i$ is a $G$-ordinary and weakly admissible filtered $\phi$-$L$-torsor.

The set of filtered $\phi$-$P/U_{i+1}$-torsors which lifts $T_i$ and admits a lifting to to a filtered $\phi$-$P$-torsor is an $\mathbb{Q}_p$-affine space isomorphic to $U_i(K\otimes E)/U_{i+1}(K\otimes E)$.

## 2.4. Crystallinity of parabolic liftings

**2.4.1. Enough gaps in Hodge-Tate weights** We say a crystalline representation $\rho : G_K \to L(E)$ has enough gaps in Hodge-Tate weights with respect to $P$ if the adjoint representation

$$G_K \xrightarrow{\rho} L(E) \to \mathrm{Aut}(\mathrm{Lie}\, U)(E)$$

has labelled Hodge-Tate weights slightly less then $\underline{0}$ in the sense of [EG19, 6.3].

We remark that having enough gaps in Hodge-Tate weights is strictly stronger than being $G$-ordinary. More precisely, $G$-ordinarity does not require one of the inequalities in [EG19, 6.3] to be strict.

**2.4.2. Proposition** A filtered $\phi$-$P$-torsor $T$ is weakly admissible if the filtered $\phi$-$L$-torsor $(\pi_L)_*(T)$ is weakly admissible with respect to $P$.

*Proof.* We first remark this proposition for general linear groups is a reformulation of the standard fact that the category of weakly admissible filtered $\phi$-modules is an abelian category (see, for example, [C11, Proposition 8.2.10]).

Write $\bar{T} = (\pi_L)_*(T)$. The parabolic $P$ of $G$ is defined by a cocharacter $\lambda$ of $G$.

Let $f : G \to \mathrm{GL}(V)$ be an algebraic representation. Then we have induced maps

$$P_G(\lambda)(f) : P = P_G(\lambda) \to P_{\mathrm{GL}(V)}(f_*\lambda) =: P'$$

and

$$L_G(\lambda)(f) : L = L_G(\lambda) \to L_{\mathrm{GL}(V)}(f_*\lambda) =: L'.$$

The filtered $\phi$-$L'$-torsor $(L_G(\lambda))_*(\bar{T})$ is weakly admissible because $\bar{T}$ is weakly admisible. Write $i$ for the embedding $P \subset G$. Since

$$(L_G(\lambda)(f))_*(\pi_L)_*T = (\pi_{L'})_*(P_G(\lambda)(f))_*T$$

using the result for $\mathrm{GL}(V)$ it follows that $i_*T$ is weakly admissible as a filtered $\phi$-$G$-torsor by standard Tannakian theory arguments. Again by Tannakian theory or more precisely the fact that the Tannakian category of representations of $P$ are generated by a single faithful embedding, $T$ is weakly admissible as a filtered $\phi$-$P$-torsor. □

Theorem 2.3.7 and Proposition 2.4.2 together give a complete description of parabolic liftings of $G$-ordinary crystalline representations, which allow us to prove Theorem (A) with the help of some Galois cohomology arguments.

**2.4.3.** Let $P$ be a parabolic of $G$ with the unipotent radical $U$ and Levi factor $L$. Let $\rho : G_K \to L(E)$ be a Galois representation. A parabolic lifting is a commutative diagram

$$\begin{array}{ccc} G_K & \xrightarrow{\quad\tilde{\rho}\quad} & P(E) \\ & {\scriptstyle\rho}\searrow \qquad \swarrow & \\ & L(E) & \end{array}$$

**Theorem** If $\rho$ is crystalline with enough gaps in Hodge-Tate weights with respect to $P$, any parabolic lifting $\tilde{\rho} : G_K \to P(E)$ is crystalline.

We'll prove the theorem by inductively constructing weakly admissible filtered $\phi$-$P/U_i$-torsors which corresponds to $\tilde{\rho}$ mod $U_i$ via Fontaine's functors $V_{\mathrm{cris}}$ and $D_{\mathrm{cris}}$.

*Proof.* Since crystallinity is insensitive to base change, we assume the filtered $\phi$-$L$-torsor $D_{\mathrm{cris}}(\rho)$ has a trivial underlying $L$-torsor, by possibly enlarging the coefficient field $E$.

By [EG19, Lemma 6.3.1], having enough gaps in Hodge-Tate weights implies for all $i$,

$$H^1_f(G_K, U_i/U_{i+1}) = H^1(G_K, U_i/U_{i+1})$$

where $H^1_f$ is the subgroup of crystalline extensions. Here $U_i/U_{i+1}$ is endowed with the adjoint action $G_K \xrightarrow{\rho} L(E) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(U_i/U_{i+1})$. Write $\rho_i$ for $\tilde{\rho}$ mod $U_i$.

We argue by induction. Assume $\rho_i : G_K \to P/U_i$ is crystalline (and admits a lifting to $P$). By Corollary 2.3.8 and Proposition 2.4.2, the set of crystalline representations $G_K \to P/U_{i+1}$ which lifts $\rho_i$ and admits a lifting to $P$ is an affine space isomorphic to

$$U_i(K \otimes E)/U_{i+1}(K \otimes E)$$

which has the same $\mathbb{Q}_p$-dimension as $H^1_f(G_K, U_i/U_{i+1})$. On the other hand, the set of all liftings (not necessarily crystalline) is an $H^1(G_K, U_i/U_{i+1})$-torsor. An injective, affine map of an affine space into another affine space

$$\begin{aligned}&\{\text{crystalline representations valued in } P/U_{i+1} \text{ lifting } \rho_i \text{ and admits a lifting to } P\}\\ \hookrightarrow&\{\text{representations valued in } P/U_{i+1} \text{ lifting } \rho_i\}\end{aligned}$$

of the same dimension is an isomorphism. By comparing the dimension, we conclude $\rho_{i+1}$ is crystalline. □

**2.4.4. Remark** We remark that proving Theorem (A) using the strategy of the proof of Proposition 2.4.2 will not work. This is because *$G$-ordinarity is not preserved by $P_G(\lambda)(f)$*. (Hint: consider the simplest example $\mathrm{Sym}^2 : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^2(V))$.)

### 2.5. Extensions of anti-$G$-ordinary filtered $\phi$-$L$-torsors

We give a more complete picture of the theory of parabolic extensions by working out the anti-$G$-ordinary case.

**2.5.1. Definition** Let $(T, \phi_T, \theta_T)$ be a filtered $\phi$-$P$-torsor. $\bar{T} = (\pi_L)_*T$ is a filtered $\phi$-$L$-torsor. We say $\theta_T$, $T$, or $\bar{T}$ is *anti-$G$-ordinary* if the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on the vector space $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})$ satisfies $\theta^1_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}(\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}) = \mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$. In other words, the Hodge polygon of $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}$ lies above the $x$-axis except for the left endpoint.

**2.5.2. Proposition** Let $(T, \phi_T, \theta_T)$ be a weakly admissible, anti-$G$-ordinary filtered $\phi$-$P$-torsor.

(1) There is a unique section $\iota : L_{K_0\otimes E} \hookrightarrow P_{K_0\otimes E}$ such that $\phi_T = \iota_*((\pi_L)_*\phi_T)$.
(2) For any section $\iota : L_{K\otimes E} \hookrightarrow P_{K\otimes E}$, $\iota_*((\pi_L)_*\theta_T) = \theta_T$.
(3) For any section $\iota : L_{K_0\otimes E} \hookrightarrow P_{K_0\otimes E}$, $(T, \phi_T, \theta_T) \cong \iota_*((\pi_L)_*(T, \phi_T, \theta_T))$.

*Proof.* (1) The proof of Lemma 2.3.6 works verbatim.

(2) We adapt the arguments of subsection 2.3.4. Choose any section $\iota : L_{K\otimes E} \hookrightarrow P_{K\otimes E}$. Let $S \subset \mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E}$ be a maximal torus. Let $S_0 \subset S$ be the maximal subtorus such that the centralizer $Z(S_0)$ is (an embedding of) the Levi factor $L_{K\otimes E}$ of $P_{K\otimes E}$. Let $U_{K\otimes E}$ be the unipotent radical of $P_{K\otimes E}$. We have the Levi decomposition

$$\mathrm{Lie}\, P_{K\otimes E} = \mathrm{Lie}\, Z(S_0) \oplus \mathrm{Lie}(U_{K\otimes E}) = \mathrm{Lie}\, Z(S_0) \oplus \bigoplus_{\alpha\in\Phi^+(S_0,G)} \mathfrak{g}_\alpha.$$

By Theorem 2.3.3, $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) = (i_*\theta_T)^0(\mathrm{Lie}\, G)$. Since $T$ is anti-$G$-ordinary, we have

$$\theta^1_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}(\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})) = \mathrm{gr}^\bullet(\mathrm{ad})(\bar{T}).$$

It is clear that the filtration $i_*\theta_T$ on $\mathrm{Lie}\, G$ and the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T}) = \mathrm{Lie}(U_{K\otimes E})$ are compatible. So $i_*\theta^0_T(\mathrm{Lie}\, G) \cap \mathrm{Lie}(U_{K\otimes E}) \supset i_*\theta^1_T(\mathrm{Lie}\, G) \cap \mathrm{Lie}(U_{K\otimes E}) = \mathrm{Lie}(U_{K\otimes E})$. Thus we have

$$(\dagger) \qquad U_{K\otimes E} \subset \mathrm{Aut}^{\otimes}(i_*\theta_T).$$

By Lemma 2.2.2, $\iota'_*((\pi_L)_*\theta_T) = \theta_T$ for some $\iota'$. There exists an $N \in U(K \otimes E)$ such that $\iota' = N\iota N^{-1}$. By (†), $N\theta_T N^{-1} = \theta_T$, and therefore $\iota_*((\pi_L)_*\theta_T) = \theta_T$.

(3) is a consequence of (1) and (2). □

**2.5.3. Corollary** Let $P$ be a parabolic of $G$ with the unipotent radical $U$ and Levi factor $L$. Let $\rho : G_K \to L(E)$ be a Galois representation. If $\rho$ is crystalline and anti-$G$-ordinary with respect to $P$, there is one and only one crystalline parabolic lifting $\tilde{\rho} : G_K \to P(E)$ of $\rho$.

*Proof.* By the previous lemma, up to isomorphism, there exists a unique parabolic extension of the filtered $\phi$-$P$-torsor which lifts $D_{\mathrm{cris}}(\rho)$. The corollary follows from the equivalence of categories explained in 2.1.7. □

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